ELEMENTI di CALCOLO delle VARIAZIONI
LEZIONE 8 - 19.3.2024

Esempio (lezione scorsa) $\qquad \mathcal{L}(u) = \frac{1}{2}\int_0^T \left((u'(x))^2 - (u(x))^2 \right) dx$

$$u(0)=0, \quad u(T)=0$$

se $T \leq \pi$ $\qquad u_0 = 0$ è minimo per $\mathcal{L}(u)$. $\mathcal{L}(0)=0$

$$\forall u \in C_0^1([0,\pi]) \qquad \int_0^\pi u^2(x) \leq \int_0^\pi (u')^2(x) \qquad (*)$$

Posso estendere questa disuguaglianza ad ogni $u \in C_0^1([0,T])$?

data $u:[0,T]\to\mathbb{R}$ $\qquad v:[0,\pi]\to\mathbb{R}$

$$v(x) = u\left(T\frac{x}{\pi}\right)$$

$$\begin{cases} v(0) = u(0) = 0 \\ v(\pi) = u(T) = 0 \end{cases}$$

$$v'(x) = \frac{T}{\pi}\cdot u'\left(T\frac{x}{\pi}\right)$$

$$\int_0^\pi v^2(x)\,dx \overset{(*)}{\leq} \int_0^\pi (v'(x))^2\,dx$$

$$\int_0^\pi u^2\left(\frac{Tx}{\pi}\right)dx = \int_0^\pi v^2(x)\,dx \qquad \int_0^\pi (v'(x))^2\,dx = \int_0^\pi \left(\frac{T}{\pi}u'\left(\frac{Tx}{\pi}\right)\right)^2 dx$$

$$\begin{cases} s = \frac{Tx}{\pi} \\ ds = \frac{T}{\pi}dx \end{cases}$$

$$\int_0^\pi u^2\left(\frac{Tx}{\pi}\right)dx = \frac{\pi}{T}\int_0^T u^2(s)\,ds \qquad \int_0^\pi \left(\frac{T}{\pi}u'\left(\frac{Tx}{\pi}\right)\right)^2 dx = \frac{T^2}{\pi^2}\cdot\frac{\pi}{T}\int_0^T (u'(s))^2\,ds$$

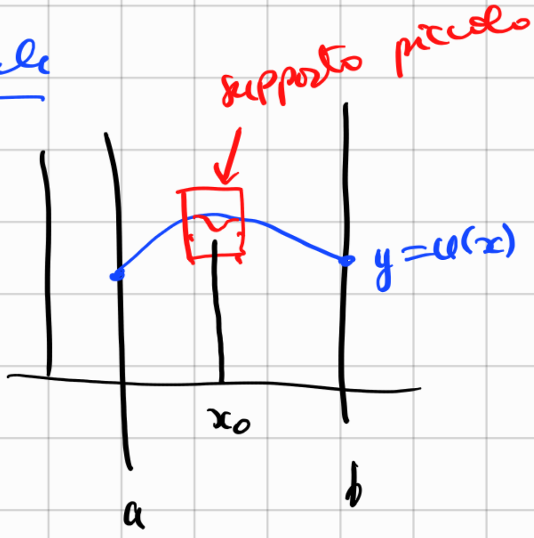
$$\int_0^T u^2(s)\,ds \leq \frac{T^2}{\pi^2}\int_0^T (u'(s))^2\,ds$$

$$\|u\|_{L^2} \le \frac{T}{\pi}\,\|u'\|_{L^2} \qquad \forall u \in C^1_0([0,T])$$

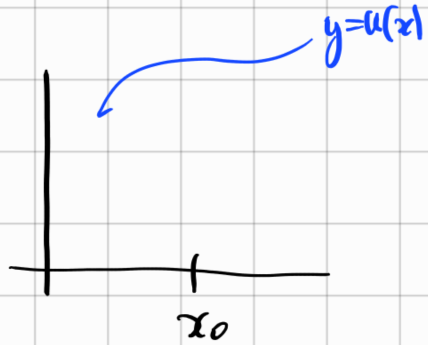
Disuguaglianza di Poincaré

Esistenza di una calibrazione locale

$z \mapsto L(x,y,z)$ convesso



① Caso scalare



Sia $u_0(x)$ una sol. di E.L.: u_0 soddisfa: $\frac{\partial L}{\partial y} = \frac{d}{dx}\frac{\partial L}{\partial z}$

Voglio costruire una fibrazione $u_t(x)$.
Esplicito E.L.:

$$\frac{\partial L}{\partial y}(x,u(x),u'(x)) = \frac{d}{dx}\frac{\partial L}{\partial z}(x,u(x),u'(x))$$
$$= \frac{\partial^2 L}{\partial x \partial z} + \frac{\partial^2 L}{\partial y \partial z}\,u'(x) + \frac{\partial^2 L}{(\partial z)^2}\cdot u''(x)$$

Supponiamo: $\frac{\partial^2 L}{(\partial z)^2} > 0$ (per poter dividere)
questo implica $z \mapsto L(x,y,z)$ è strettamente convessa.

$$u''(x) = \frac{\frac{\partial L}{\partial y}(x,u,u') - \frac{\partial^2 L}{\partial x\partial z}(\dots) - \frac{\partial^2 L}{\partial y \partial z}(\dots)\cdot u'(x)}{\frac{\partial^2 L}{(\partial z)^2}(\dots)}$$

Eq. in forma normale. Voglio impostare un problema di Cauchy:

$$\begin{cases} u_t'' = \dfrac{\frac{\partial L}{\partial y}(\dots) - \frac{\partial^2 L}{\partial x \partial z}(\dots) - \frac{\partial^2 L}{\partial y \partial z}(\dots)\cdot u_t'}{\frac{\partial^2 L}{\partial z^2}(\dots)} \\ u_t(x_0) = u_0(x_0) + t \\ u_t'(x_0) = u_0'(x_0) \end{cases}$$

Per Cauchy-Lipschitz $\exists \delta$ (indipendente da t) per cui le soluzioni u_t esistono e sono definite su $[x_0-\delta, x_0+\delta]$.

Devo verificare che u_t sono una fibrazione.

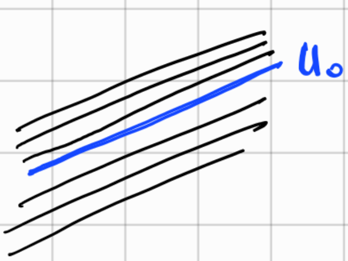
$$f(x,t) = \begin{pmatrix} x \\ u_t(x) \end{pmatrix}$$ deve essere invertibile.

Basta verificare che $Df(x_0,0)$ è invertibile. Se ciò succede f è invertibile in un intorno di $(x_0,0)$ per il teorema di invertibilità locale.

$$\frac{\partial f}{\partial x}(x,t) = \begin{pmatrix} 1 \\ u_t'(x) \end{pmatrix}, \quad \frac{\partial f}{\partial t}(x_0,t) = \begin{pmatrix} \frac{d}{dt} x_0 \\ \frac{d}{dt}(u_0(x_0)+t) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$Df(x_0,0) = \begin{pmatrix} 1 & 0 \\ u_0'(x_0) & 1 \end{pmatrix}$$ $\det = 1$ è invertibile!

Es (geodetiche) $\quad \mathcal{L}(u) = \int_a^b \sqrt{1+(u')^2}$



$u_0(x) = mx + q$

$u_t(x) = mx + q + t$

② Caso vettoriale.

Dobbiamo vedere il metodo delle calibrazioni.

$$\Omega = \left(L - \frac{\partial L}{\partial z}\cdot z\right)dx + \frac{\partial L}{\partial z}\, dy$$

nel caso scalare.

nel caso vettoriale

$$\Omega = \left(L - \sum_k \frac{\partial L}{\partial z_k}\cdot z_k\right)dx + \sum_k \frac{\partial L}{\partial z_k}\, \underbrace{dy_k}$$

$\underline{z} = u_t'(x) = \frac{\partial g}{\partial x}$

$\underline{y} = \underline{u}_t(x) = \underline{g}(x,t)$

$$y_k = g_k(x,t) \qquad dy_k = \frac{\partial g_k}{\partial x}\cdot dx + \sum_j \frac{\partial g_k}{\partial t_j}\cdot dt_j$$

$$(\Phi\circ f)_* \Omega = \left(L - \cancel{\sum_k \frac{\partial L}{\partial z_k}\overset{z}{\frac{\partial g_k}{\partial x}}}\right)dx + \sum_k \frac{\partial L}{\partial z_k}\left(\cancel{\frac{\partial g_k}{\partial x}dx} + \sum_j \frac{\partial g_k}{\partial t_j}dt_j\right)$$

$$= L\, dx + \sum_j \sum_k \frac{\partial L}{\partial z_k}\frac{\partial g_k}{\partial t_j}\, dt_j$$

$$= \underbrace{L}\, dx + \underbrace{\sum_j \left(\nabla_z L \cdot \frac{\partial g}{\partial t_j}\right)}\underbrace{dt_j}$$

è chiusa?

$$\boxed{EL: \nabla_y L = \frac{d}{dx}\nabla_z L}$$

① $$\frac{d}{dt_j} L(x, u_t(x), u_t'(x)) \overset{?}{=} \frac{d}{dx}\left(\nabla_z L \cdot \frac{\partial g}{\partial t_j}\right)$$

ok

$u_t(x) = g$, $u_t'(x) = \frac{\partial g}{\partial x}$

$$\nabla_y L \cdot \frac{\partial g}{\partial t_j} + \nabla_z L \cdot \frac{\partial^2 g}{\partial x \partial t_j} \overset{!}{=} \nabla_y L \cdot \frac{\partial g}{\partial t_j} + \nabla_z L \cdot \frac{\partial^2 g}{\partial t_j \partial x}$$

② $$\boxed{\frac{d}{dt_j}\left(\nabla_z L \cdot \frac{\partial g}{\partial t_i}\right) = \frac{d}{dt_i}\left(\nabla_z L \cdot \frac{\partial g}{\partial t_j}\right)}$$

$$[t_i, t_j] \doteq \frac{d}{dt_j}\left(\nabla_z L \cdot \frac{\partial g}{\partial t_i}\right) - \frac{d}{dt_i}\left(\nabla_z L \cdot \frac{\partial g}{\partial t_j}\right)$$

↑

Parentesi di Lagrange.

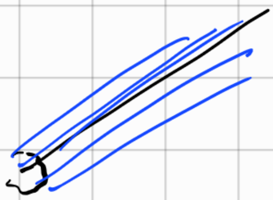
Dobbiamo mettere come ipotesi $\boxed{[t_i, t_j] = 0}$.

Es (geodetiche vettoriali) $\quad \mathcal{L}(\underline{u}) = \int_0^1 |\underline{u}'(x)|\,dx$

$$u_t(x) = \underline{m}\,x + \underline{q} + \underline{t} \qquad u_0(x) = \underline{m}\,x + q$$

$$|\underline{m}| = 1$$

$$L(x, \underline{y}, \underline{z}) = |\underline{z}|$$



$$\nabla_z L = \frac{z}{|z|} \qquad \underline{g}(x,t) = \underline{m}\,x + \underline{q} + \underline{t}$$

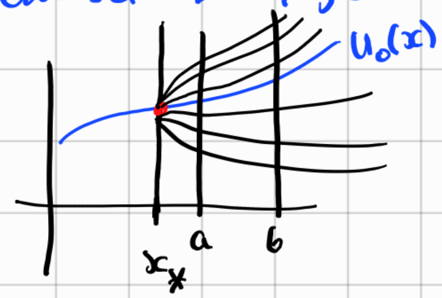
$$\frac{d}{dt_j} \nabla_z L \cdot \frac{\partial g}{\partial t_i} = \frac{d}{dt_j}\left(\underline{m} \cdot e_i\right) = 0$$

ok

Costruiamo una fibrazione locale
(caso vettoriale)

il metodo precedente non garantisce $[t_i, t_j] = 0$

Metodo alternativo



Eq. di E.L. $\quad \nabla_y L = \frac{d}{dx} \nabla_z L(x, u_t(x), \underbrace{u_t'(x)})$

espandiamo: $\quad \nabla_y L = \nabla_z \frac{\partial L}{\partial x} + D^2_{zy} L \cdot u_t'(x) + D^2_{zz} L \cdot \underbrace{u_t''(x)}$

Ipotesi $D^2_{zz} L(x_*, u_0(x_*), u_0'(x_*))$ sia invertibile

$\Rightarrow$ invertibile anche in un intorno di $(x_*, u_0(x_*), u_0'(x_*))$.

$$\begin{cases} u_t''(x) = \left(D^2_{zz} L\right)^{-1} \cdot \left(\nabla_y L - \nabla_z \frac{\partial L}{\partial x} - D^2_{zy} L\, u_t'(x)\right) \\ u_t(x_*) = u_0(x_*) \\ u_t'(x_*) = u_0'(x_*) + t \end{cases}$$

Ci sarà una soluzione unica locale, che mi darà una fibrazione. $\quad$ (*) VEDI PAGINA SEGUENTE

Verifichiamo se $[t_i, t_j] = 0$.

Idea: ① $[t_i, t_j](x_*) = 0$, ② $\frac{d}{dx}[t_i, t_j] = 0$

$$[t_i, t_j] = \frac{\partial}{\partial t_i}\left(\nabla_z L \cdot \frac{\partial g}{\partial t_j}\right) - \frac{\partial}{\partial t_j}\left(\nabla_z L \cdot \frac{\partial g}{\partial t_i}\right)$$

$g(x,t) = u_t(x)$

① $g(x_*, t) = u_t(x_*) = u_0(x_*)$ non dipende da t

$$\frac{\partial g}{\partial t_j}(x_*, t) = 0 \quad \Rightarrow \quad [t_i, t_j] = 0$$

② $$\frac{d}{dx}[t_i, t_j] = \underbrace{\frac{d}{dx}\frac{\partial}{\partial t_i}\left(\nabla_z L \cdot \frac{\partial g}{\partial t_j}\right)} - \frac{d}{dx}\frac{\partial}{\partial t_j}\left(\nabla_z L \cdot \frac{\partial g}{\partial t_i}\right)$$

(*) è simmetrico?

$$(*) = \frac{\partial}{\partial t_i}\frac{d}{dx}\left(\nabla_z L \cdot \frac{\partial g}{\partial t_j}\right) \underset{EL}{=} \frac{\partial}{\partial t_i}\left(\nabla_y L \cdot \frac{\partial g}{\partial t_j} + \nabla_z L \cdot \frac{\partial^2 g}{\partial t_j \partial x}\right)$$

$$L = L(x, u_t, u_t') = L\left(x, g, \frac{\partial g}{\partial x}\right)$$

$$= D^2_{yy} L \cdot \frac{\partial g}{\partial t_i} \cdot \frac{\partial g}{\partial t_j} + D^2_{yz} L \cdot \frac{\partial^2 g}{\partial x \partial t_i} \cdot \frac{\partial g}{\partial t_j} + \nabla_y L \frac{\partial^2 g}{\partial t_i \partial t_j} +$$

$$+ D^2_{yz} L \cdot \frac{\partial g}{\partial t_i} \cdot \frac{\partial^2 g}{\partial t_j \partial x} + D^2_{zz} L \cdot \frac{\partial^2 g}{\partial x \partial t_i} \cdot \frac{\partial^2 g}{\partial t_j \partial x}$$

$$+ \nabla_z L \cdot \frac{\partial^3 g}{\partial t_i \partial t_j \partial x} \overset{?}{=} \text{simmetrico rispetto } (i,j)$$

$\square$

AGGIUNTO A FINE LEZIONE

$(*)$

$$f(x,t) = \begin{pmatrix} x \\ u_t(x) \end{pmatrix}$$

per $x \to x_*$

$$u_t(x) = u_t(x_*) + (x - x_*)\, u_t'(x_*) + o(x - x_*)$$

$$= u_0(x_*) + (x - x_*)\left(u_0'(x_*) + t\right) + o(x - x_*)$$

$$\frac{\partial}{\partial t_k} u_t(x) = (x - x_*) \cdot e_k + o(x - x_*)$$

$$Df(x, 0) = \left(\begin{array}{c|c} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial t_k} \\ \hline u_t'(x) & \frac{\partial u_t(x)}{\partial t_k} \end{array}\right) \sim \left(\begin{array}{c|c} 1 & 0 \\ \hline u_t'(x) & (x - x_*) \cdot I_d \end{array}\right)$$

$=$ invertibile per $x \to x_*$