ELEMENTI di CALCOLO DELLE VARIAZIONI
LEZIONE 2 — 27.2.2024

testo di riferimento: Appunti di Calcolo delle Variazioni
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Ieri: Funzionale classico: $\mathcal{I}(u) = \int_a^b L(x, u(x), u'(x))\,dx$ $\quad L = L(x,y,z)$

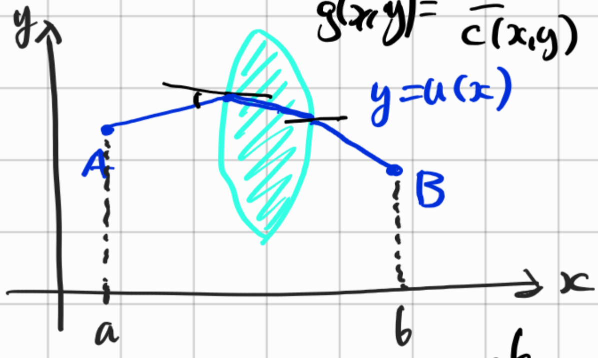
Variazione prima: $\delta\mathcal{I}$

Equazione di Eulero-Lagrange: $\boxed{\dfrac{\partial L}{\partial y} = \dfrac{d}{dx}\dfrac{\partial L}{\partial z}}$

Esempio brachistocrona.

Oggi altri esempi.

1. il problema di Fermat



Minima azione:

$$\mathcal{I}(u) = \int_a^b g(x, u(x)) \cdot \sqrt{1 + (u'(x))^2}\,dx = \int_a^b L(x, u(x), u'(x))\,dx$$

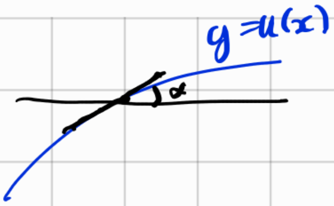
u minimizza $\mathcal{I}$ tra le curve con estremi A, B fissati.

$$L(x,y,z) = g(x,y) \cdot \sqrt{1+z^2} \qquad \begin{cases} \dfrac{\partial L}{\partial y} = \dfrac{\partial g}{\partial y}\sqrt{1+z^2} \\ \dfrac{\partial L}{\partial z} = g(x,y)\dfrac{z}{\sqrt{1+z^2}} \end{cases}$$

$$EL: \frac{\partial g}{\partial y}(x, u(x)) \cdot \sqrt{1+(u')^2} = \frac{d}{dx}\left(\boxed{g(x, u(x)) \cdot \frac{u'(x)}{\sqrt{1+(u'(x))^2}}}\right)$$

① se g è costante: $\quad 0 = \dfrac{d}{dx}\dfrac{u'(x)}{\sqrt{1+(u'(x))^2}}$

$$\frac{u'(x)}{\sqrt{1+(u'(x))^2}} = \text{costante} \qquad u' \text{ è costante.}$$



$$\operatorname{tg}\alpha = u'(x) \qquad 1+\operatorname{tg}^2 x = \frac{1}{\cos^2 x}$$

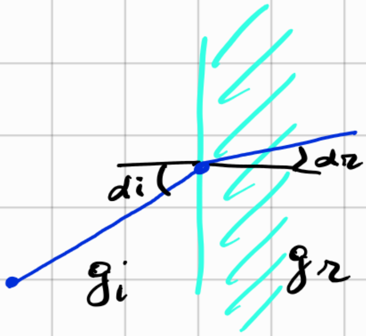
$$\sin^2 x = \operatorname{tg}^2 x \cdot \cos^2 x \, , \quad \frac{1}{1+\operatorname{tg}^2 x} = \cos^2 x$$

$$= \frac{\operatorname{tg}^2 x}{1+\operatorname{tg}^2 x}$$

$$\sin\alpha = \frac{u'(x)}{\sqrt{1+(u'(x))^2}}$$

$u(x)$ è lineare.

(2)



i = incidenza

r = rifrazione

$g = g(x)$

$$\frac{\partial g}{\partial y} = 0$$

$$g(x, u(x)) \cdot \sin\alpha(x) = \text{costante}$$

$$g_i \cdot \sin\alpha_i = g_r \cdot \sin\alpha_r$$

$$\boxed{\frac{\sin\alpha_i}{\sin\alpha_r} = \frac{g_r}{g_i}}$$

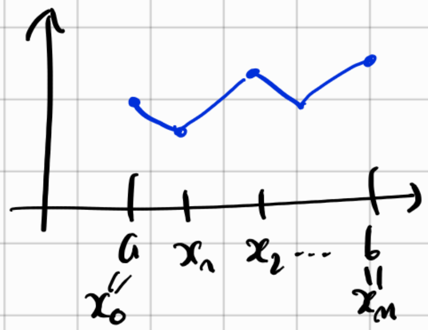
LEGGE di SNELL

In questo tipo di problemi ha senso considerare

$u \in C^1_p([a,b]) = C^1$ "a tratti" $\qquad u \in C^0([a,b])$

$a = x_0, x_1, \dots x_n = b \in [a,b]$

tali che $u \in C^1([x_{k-1}, x_k])$



FORMA VETTORIALE

$\underline{u} : [a,b] \to \mathbb{R}^n$

$$\mathcal{L}(\underline{u}) = \int_a^b L(x, \underline{u}(x), \underline{u}'(x))\,dx$$

$\underline{u}(x) = (u_1(x), \dots, u_n(x))$

$\underline{u}'(x) = (u_1'(x), \dots, u_n'(x))$

$L = L(x, \underline{y}, \underline{z})$

$L : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$\left[\frac{d}{d\varepsilon} \mathcal{L}(\underline{u} + \varepsilon \underline{\varphi})\right]_{\varepsilon=0} = \int_a^b \frac{d}{d\varepsilon} L(x, \underline{u} + \varepsilon\underline{\varphi}, \underline{u}' + \varepsilon\underline{\varphi}')\,dx \Big|_{\varepsilon=0}$$

$L = L(x, y_1, \dots, y_n, z_1, \dots, z_n)$

$$= \int_a^b \Big[\sum_{k=1}^n \frac{\partial L}{\partial y_k} \cdot \varphi_k + \underbrace{\sum_{k=1}^n \frac{\partial L}{\partial z_k} \cdot \varphi_k'}\Big]\,dx$$

$$= \int \Big[\nabla_y L(x, \underline{u}, \underline{u}') \cdot \underline{\varphi} - \frac{d}{dx} \nabla_z L(x, u, u') \cdot \varphi \Big]\,dx$$

E.L. $\quad \boxed{\nabla_y L = \frac{d}{dx} \nabla_z L}$

Problema di Fermat con le curve

$\underline{u}(a) = A$

$\underline{u}(b) = B$

$$\mathcal{L}(\underline{u}) = \int_a^b g(\underline{u}(x)) \cdot |u'(x)|\,dx$$

$$= \int_a^b L(x, \underline{u}(x), \underline{u}'(x))\,dx$$

$$L(x, \underline{y}, \underline{z}) = g(\underline{y}) \cdot |\underline{z}|$$

$$\begin{cases} \nabla_y L = \nabla g(\underline{y}) \cdot |z| \\ \nabla_z L = g(\underline{y}) \cdot \dfrac{z}{|z|} \end{cases}$$

E.L.: $\nabla g(\underline{u}(x))\cdot|\underline{u}'(x)| = \frac{d}{dx}\left(g(\underline{u}(x))\cdot\frac{u'(x)}{|u'(x)|}\right)$

Supponiamo che u sia un minimo di $\mathcal{L}$ con estremi fissati A, B

supponiamo inoltre che $|u'(x)|=1$

(parametrizzato per lunghezza d'arco)

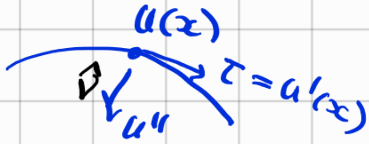
$$(*) \quad \boxed{\nabla g(u(x)) = \frac{d}{dx}\, g(u(x))\cdot u'(x)}$$

① Moltiplico scalarmente per il versore $\hat n$: $\quad \hat n \perp \nabla g$

$$0 = \frac{d}{dx}\, g(u(x))\,(u'(x)\cdot\hat n)$$

$$g(u(x))\cdot\sin\alpha(x) = \text{costante}.$$

② Moltiplico scalarmente per $\hat\nu$



$|u'|=1 \qquad |u'|^2=1$

$$0 = \frac{d}{dx}(u')^2 = 2u'\cdot u'' \qquad u''\perp u'$$

$$u'' = k\cdot\underline{\nu}$$

curvatura

(*) scalar $\hat\nu$

$$\nabla g(u(x))\cdot\hat\nu = \hat\nu\cdot\frac{d}{dx}\left(g(u(x))\cdot u'(x)\right)$$

$$\parallel$$

$$\frac{\partial g}{\partial\nu}$$

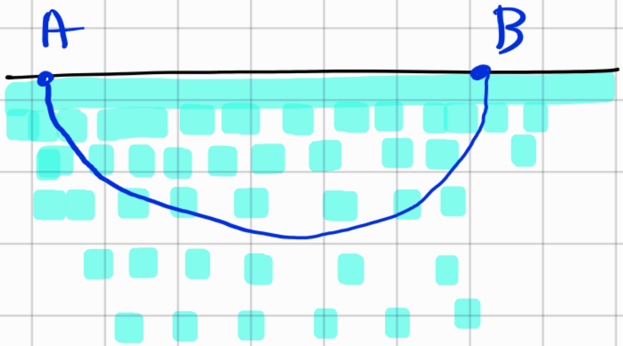
$$= \hat\nu\cdot\left(\nabla g\cdot u'\cdot u'(x) + g\cdot u''(x)\right)$$

$$= (\nabla g\cdot u')\,\cancel{\hat\nu\cdot u'} + g\cdot\underbrace{u''\cdot\hat\nu}_{k_u}$$

$$g \cdot k = \frac{\partial g}{\partial \nu}$$

$\uparrow$ curvatura.

Oss la brachistocrona è un caso particolare del problema di Fermat con densità $g(x,y) = \frac{1}{\sqrt{x}}$ $\quad \int \frac{\sqrt{1+u'^2}}{\sqrt{x}}$



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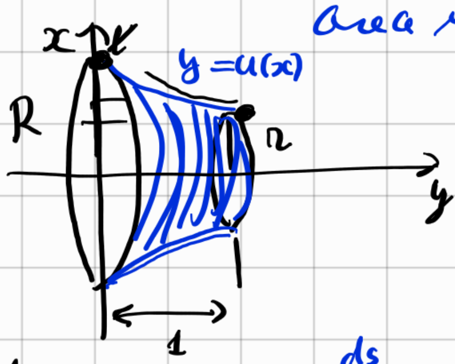
2. CATENOIDE

area minima



$u : [r, R] \to \mathbb{R}$

$$\mathcal{L}(u) = 2\pi \int_r^R x \cdot \sqrt{1+(u'(x))^2}\, dx$$

$u(r) = 1 \qquad u(R) = 0$

$$= 2\pi \int_r^R L(x, u(x), u'(x))\, dx$$

$$L(x, y, z) = x \cdot \sqrt{1+z^2}$$

$$\frac{\partial L}{\partial y} = 0$$

$$\frac{\partial L}{\partial z} = x \cdot \frac{z}{\sqrt{1+z^2}}$$

EL $\quad \frac{\partial L}{\partial z}(x, u, u') = \text{costante}$

$$x \cdot \frac{u'}{\sqrt{1+(u')^2}} = c$$

$$x\, u' = c\sqrt{1+(u')^2}$$

$$x^2 (u')^2 = c^2\left(1+(u')^2\right)$$

$$(x^2-c^2)(u')^2 = c^2 \qquad (u')^2 = \frac{c^2}{x^2-c^2}$$

$$(c<0 \dots)$$

$$u' = -\frac{c}{\sqrt{x^2-c^2}} = \frac{|c|}{\sqrt{x^2-c^2}}$$

$$u = \int \frac{|c|}{\sqrt{x^2-c^2}}\,dx = \int \frac{1}{\sqrt{\left(\frac{x}{|c|}\right)^2-1}}\,dx$$

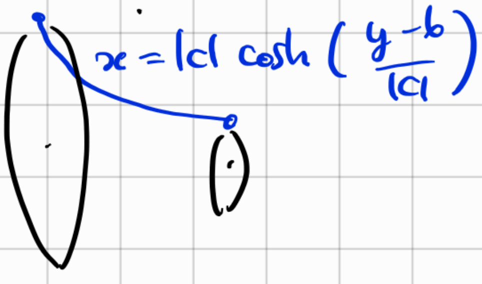
$$\boxed{\cosh^2 s - \sinh^2 s = 1}$$

$$\cosh x = \frac{e^x+e^{-x}}{2} \qquad \sinh x = \frac{e^x-e^{-x}}{2}$$

$$\operatorname{settcosh}' s = \frac{1}{\sinh(\operatorname{settcosh} s)} = \frac{1}{\sqrt{s^2-1}}$$

$$y = u(x) = |c| \cdot \operatorname{settcosh}\left(\frac{x}{|c|}\right) + b$$

$$\cosh\left(\frac{y-b}{|c|}\right) = \frac{x}{|c|} \qquad x = |c|\cosh\left(\frac{y-b}{|c|}\right)$$



ES.3 Meccanica lagrangiana

$$E = \frac{1}{2}mv^2$$

$$U = U(u(x))$$

$$F = -\nabla U$$

$$\mathcal{L}(u) = \int L(x, u, u')\,dx = \int \frac{1}{2}m(u')^2 - U(u(x))$$

$$L(x,y,z) = \frac{1}{2}mz^2 - U(y)$$

$$\frac{\partial L}{\partial y} = -\nabla U = F$$

$$\frac{\partial L}{\partial z} = mz$$

E.L: $-\nabla U(u(x)) = \frac{d}{dx} m u'(x)$

$$F = m \cdot a$$

conservazione dell'energia

$$-\nabla U(u(x)) = m\, u''(x)$$

moltiplico per $u'(x)$

$$-u' \cdot \nabla U(u(x)) = m\, u'' \cdot u'$$

$$-\frac{d}{dx} U(u(x)) = \frac{d}{dx} \frac{1}{2} m (u')^2$$

$$\frac{d}{dx}\left[U(u(x)) + \frac{1}{2} m (u')^2\right] = 0$$