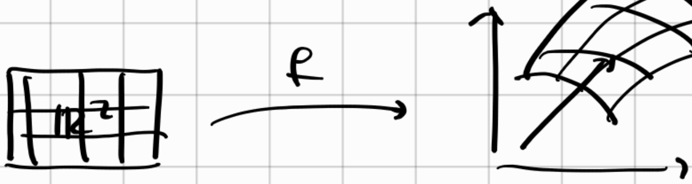
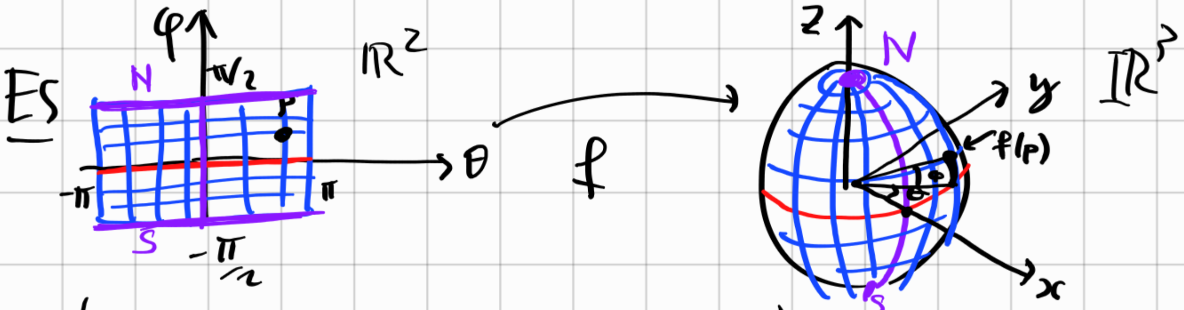
ANALISI MATEMATICA
17.3.2023

$$f:\mathbb{R}^n \to \mathbb{R}^m \quad \begin{cases} f:\mathbb{R}^n \to \mathbb{R} & m=1 \\ f:\mathbb{R} \to \mathbb{R}^m & n=1 \end{cases}$$

Es: $f:\mathbb{R}^2 \to \mathbb{R}^3$



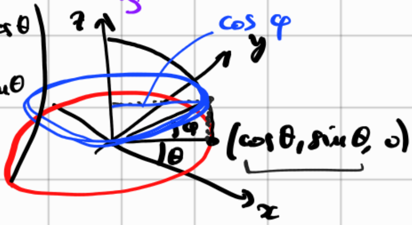
Es



longitudine

θ φ

$$f(\theta,\varphi) = \begin{pmatrix} x = \cos\varphi \cdot \cos\theta \\ y = \cos\varphi \cdot \sin\theta \\ z = \sin\varphi \end{pmatrix}$$



$$f:[-\pi,\pi]\times\left[-\tfrac{\pi}{2},\tfrac{\pi}{2}\right] \to \mathbb{R}^3$$

Formula di Taylor al I ordine:

se $f:\Omega \subseteq \underset{\underline{x}}{\mathbb{R}^n} \to \underset{\underline{y}}{\mathbb{R}^m}$ è di classe C^1:

$$\underset{\mathbb{R}^m}{\underline{y}} = f(\underline{x}) = \underbrace{f(\underline{x}_0)}_{\mathbb{R}^m} + \underbrace{Df(\underline{x}_0)}\cdot\underbrace{(\underline{x}-\underline{x}_0)}_{\mathbb{R}^n} + o(|\underline{x}-\underline{x}_0|) \quad \text{per } \underline{x}\to\underline{x}_0$$

$$Df(\underline{x}_0) = \left(\frac{\partial f_j}{\partial x_k}\right)_{m\times n} \qquad \begin{array}{l} k = 1 \dots n \\ j = 1 \dots m \end{array}$$

$$\underline{f}(\underline{x}) = \begin{pmatrix} f_1(\underline{x}) \\ \vdots \\ f_m(\underline{x}) \end{pmatrix}$$

$$Df(\underline{x}_0) = \begin{pmatrix} Df_1 \\ \hline Df_2 \\ \hline \vdots \\ \hline Df_m \end{pmatrix} \updownarrow m \qquad \underleftrightarrow{\quad n \quad}$$

$$Df_1 = (\nabla f_1)^t \qquad \nabla f_1 = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} \\ \vdots \\ \frac{\partial f_1}{\partial x_n} \end{pmatrix}$$

$\boxed{\text{MATRICE JACOBIANA}}$ $\quad Jf(x_0)$

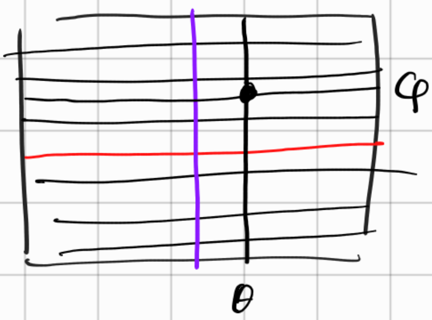
Es $\quad \underline{f}(\theta,\varphi) = \begin{pmatrix} \cos\varphi\cdot\cos\theta \\ \cos\varphi\cdot\sin\theta \\ \sin\varphi \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$ $\quad$ 3 righe

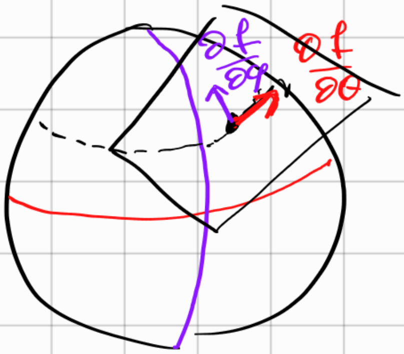
$$Df(\theta,\varphi) = \begin{pmatrix} \frac{\partial f_1}{\partial \theta} & \frac{\partial f_1}{\partial \varphi} \\ \frac{\partial f_2}{\partial \theta} & \frac{\partial f_2}{\partial \varphi} \\ \frac{\partial f_3}{\partial \theta} & \frac{\partial f_3}{\partial \varphi} \end{pmatrix} = \left(\frac{\partial \underline{f}}{\partial \theta} \,\middle|\, \frac{\partial \underline{f}}{\partial \varphi} \right) \leftarrow \text{3 righe}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \end{array}$$

2 colonne

$$\text{nel nostro esempio} = \begin{pmatrix} \underbrace{\boxed{\begin{matrix} -\cos\varphi\cdot\sin\theta \\ \cos\varphi\cdot\cos\theta \\ 0 \end{matrix}}}_{\frac{\partial f}{\partial \theta}} & \overbrace{\boxed{\begin{matrix} -\sin\varphi\cdot\cos\theta \\ -\sin\varphi\,\sin\theta \\ \cos\varphi \end{matrix}}}^{\frac{\partial f}{\partial \varphi}} \end{pmatrix}$$





Se $\quad \gamma : \mathbb{R} \to \mathbb{R}^3 \qquad \gamma(t) = \gamma(t_0) + \gamma'(t_0)\cdot(t - t_0) + o(|t-t_0|)$

Casi particolari: $\qquad$ per $x \to x_0$

prodotto tra numeri reali

$n=1,\ m=1 \qquad f(x) = f(x_0) + f'(x_0)\cdot(x - x_0) + o(|x - x_0|)$

prodotto scalare tra vettori di $\mathbb{R}^n$

$n>1,\ m=1 \qquad f(\underline{x}) = f(\underline{x}_0) + \nabla f(x_0)\cdot(\underline{x} - \underline{x}_0) + o(|x - x_0|)$.

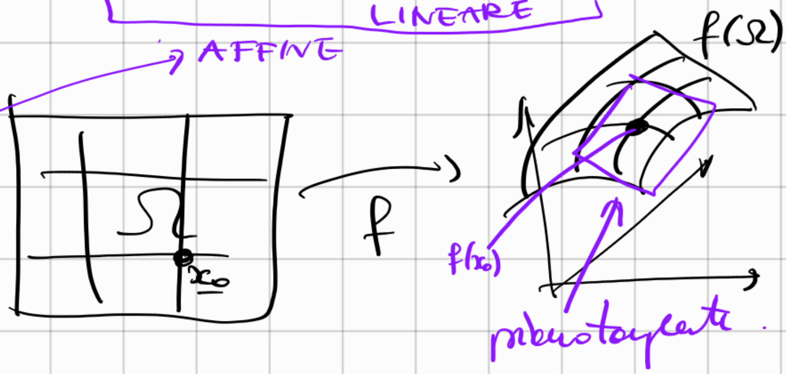
$n=1,\ m>1$ $\quad \underline{r}(t) = \underline{r}(t_0) + \underline{r}'(t_0)\cdot(t-t_0) + o(|t-t_0|)$

prodotto "per scalare".

$n>1,\ m>1$ caso generale $\quad \underline{f}(\underline{x}) = \underline{f}(\underline{x}_0) + \underbrace{\underbrace{D\underline{f}(x_0)\cdot(\underline{x}-\underline{x}_0)}_{\text{LINEARE}}}_{\text{AFFINE}} + o(|x-x_0|)$.

Se $\underline{f}: \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$

PIANO TANGENTE!



Il piano tangente alla superficie $f(\Omega)$ nel punto $f(x_0)$ è

$$\underline{f}(\underline{x}_0) + \operatorname{Im} Df(x_0)$$

spazio generato dalle colonne

$$\operatorname{Im} Df(x_0) = \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$$

Se $f: \mathbb{R}^n \to \mathbb{R}$ $\quad (m=1)$

Abbiamo una **formula di Taylor al II ordine:** $\quad f \in C^2$.

prodotto scalare

$$f(\underline{x}) = \underbrace{f(\underline{x}_0)}_{\mathbb{R}} + \underbrace{\nabla f(x_0)}_{\text{vettore } \mathbb{R}^n} \cdot \underbrace{(\underline{x}-\underline{x}_0)}_{\text{vettore } \mathbb{R}^n} + \frac{1}{2}\boxed{\underbrace{(x-x_0)^t}_{1\times n} \cdot \underbrace{D^2 f(x_0)}_{n\times n} \cdot \underbrace{(x-x_0)}_{n\times 1} = \underset{1\times 1}{\square}} + o(|x-x_0|^2)$$

FORMA QUADRATICA

$$Hf(x_0) = D^2 f(x_0) = \left(\frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_k} \right)_{n\times n} \qquad \frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_k} = \frac{\partial^2 f}{\partial x_j \partial x_k}$$

matrice Hessiana.

Esempio $\quad f(x,y,z) = x^2 + z\cdot y^2$

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} 2 \cdot x \\ 2 \cdot z \cdot y \\ y^2 \end{pmatrix}$$

$$D^2 f = \begin{pmatrix} \frac{\partial^2 f}{(\partial x)^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{(\partial y)^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{(\partial z)^2} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2z & 2y \\ 0 & 2y & 0 \end{pmatrix}$$

$$f = x^2 + zy^2 = f(0,0,0) + \nabla f(0,0,0) \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \frac{1}{2}(x,y,z) D^2 f(0,0,0) \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + o(x^2+y^2+z^2)$$

$$\begin{pmatrix} x_0 = 0 \\ y_0 = 0 \\ z_0 = 0 \end{pmatrix}$$

$$= 0 + 0 + \frac{1}{2}(x,y,z) \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}} + o(x^2+y^2+z^2)$$

$$= \frac{1}{2}(x,y,z) \cdot \begin{pmatrix} 2x \\ 0 \\ 0 \end{pmatrix} + o(x^2+y^2+z^2)$$

$$= \frac{1}{2} \cdot 2x^2 + o(x^2+y^2+z^2) = x^2 + o(x^2+y^2+z^2)$$

In effetti $zy^2 = o(x^2+y^2+z^2)$ infatti $\lim_{(x,y,z)\to(0,0,0)} \frac{zy^2}{x^2+y^2+z^2} = 0$

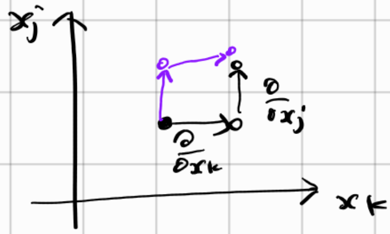
$$\left.\begin{aligned} |z| &= \sqrt{z^2} \le \sqrt{x^2+y^2+z^2} \\ |y| &= \sqrt{y^2} \le \sqrt{x^2+y^2+z^2} \end{aligned}\right| \Rightarrow |z \cdot y^2| \le \left(\sqrt{x^2+y^2+z^2}\right)^3$$

$$\left| \frac{zy}{x^2+y^2+z^2} \right| \le \left(\sqrt{x^2+y^2+z^2}\right)^{3-2} = \sqrt{x^2+y^2+z^2} \longrightarrow 0$$

Teorema (Schwartz) Se $f \in C^2$ allora:

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_k}\right) = \frac{\partial}{\partial x_k}\left(\frac{\partial f}{\partial x_j}\right)$$

ovvero $Hf = D^2 f$ è simmetrica.



Esempio $f(x,y) = x^2 \sin y$

$$\frac{\partial^2 f}{\partial y \partial x} \overset{?}{=} \frac{\partial^2 f}{\partial x \partial y}$$

$$\begin{cases} \frac{\partial f}{\partial x} = 2x \cdot \sin y \\ \frac{\partial f}{\partial y} = x^2 \cdot \cos y \end{cases}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2x \sin y) = 2x \cdot \cos y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial}{\partial x}(x^2 \cos y) = 2x \cdot \cos y \, .$$

Esempio $f(x,y) = x^3 e^{x+y} - y^2 \sqrt{x}$

$$\begin{cases} \frac{\partial f}{\partial x} = 3x^2 \cdot e^{x+y} + x^3 \cdot e^{x+y} - \frac{y^2}{2\sqrt{x}} = (3x^2 + x^3)e^{x+y} - \frac{y^2}{2\sqrt{x}} \\ \frac{\partial f}{\partial y} = x^3 \cdot e^{x+y} - 2y \cdot \sqrt{x} \end{cases}$$

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = (3x^2 + x^3) \cdot e^{x+y} - \frac{y}{\sqrt{x}}$$

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = 3x^2 \cdot e^{x+y} + x^3 \cdot e^{x+y} - \frac{y}{\sqrt{x}} = (3x^2 + x^3)e^{x+y} - \frac{y}{\sqrt{x}} \, .$$

RICERCA DI MASSIMI E MINIMI
(OTTIMIZZAZIONE)

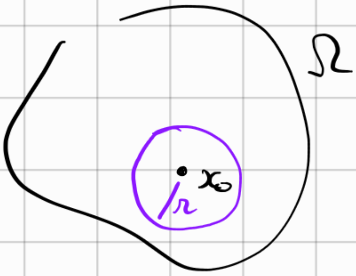
$f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$

Un punto $x_0 \in \Omega$ si dice punto di massimo (minimo) se

$f(x_0) \geq f(x) \quad \forall x \in \Omega$
$(\leq)$

x_0 si dice punto di massimo locale (o relativo)
se esiste un $r > 0$

t.c. $f(x_0) \geq f(x) \quad \forall x \in \Omega : |x - x_0| < r$
$(\leq)$



Criteri per determinare tali punti.

Criterio al I ordine: Se $f \in C^1$ e se $x_0 \in \Omega$ è un punto INTERNO a Ω e se x_0 è un massimo o un minimo relativo allora:

$$\boxed{\nabla f(x_0) = \underline{0}}$$

x_0 è un punto critico.

dim So ad esempio x_0 è un punto di massimo:

Taylor: $f(x) = f(x_0) + \nabla f(x_0) \cdot (x - x_0) + o(|x - x_0|)$

$f(x) \leq f(x_0) \quad \forall x$ t.c. $|x - x_0| < r$.

$$0 \geq f(x) - f(x_0) = \nabla f(x_0) \cdot \underbrace{(x - x_0)} + o(\underbrace{|x - x_0|})$$

se pongo $x = \boxed{x_0 + t \cdot \nabla f(x_0)}$ $\quad t > 0$

$x - x_0 = t \cdot \nabla f(x_0)$

$$= \nabla f(x_0) \cdot t \nabla f(x_0) + o(t |\nabla f(x_0)|)$$

$$= t \cdot |\nabla f(x_0)|^2 + o(t)$$

$$= t \left[\underbrace{|\nabla f(x_0)|^2} + o(1) \right] \qquad t > 0$$

se $\nabla f(x_0) \neq 0 \Rightarrow |\nabla f(x_0)| > 0$

per $t > 0$ abbastanza piccolo è ≥ 0.

<u>Esempio</u> Sia $f(x,y) = xy + x^2 + y^4$ $\qquad f: \mathbb{R}^2 \to \mathbb{R}$

Determinare, se esistono i punti di massimo e minimo di questa funzione.

Si vede che non c'è un massimo assoluto: $f(x,0) = x^2 \to +\infty$ se $x \to \pm\infty$

$\sup f = +\infty$

[in realtà $\lim_{\sqrt{x^2+y^2} \to \infty} f = +\infty$.]

Ha minimo? (Intuitivamente sì, per Weierstraß)

Se (x_0, y_0) è punto di minimo

allora $\nabla f(x_0, y_0) = \underline{0}$ (è un punto critico)

$$f = xy + x^2 + y^4 \qquad \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} y + 2x \\ x + 4y^3 \end{pmatrix}$$

$$\nabla f = \underline{0} \iff \begin{cases} y + 2x = 0 \\ x + 4y^3 = 0 \end{cases}$$

$$\begin{cases} y = -2x \\ x + 4(-2x)^3 = 0 \end{cases} \qquad \begin{cases} y = -2x \\ x - 32 \cdot x^3 = 0 \end{cases}$$

$$\boxed{f = xy + x^2 + y^4}$$

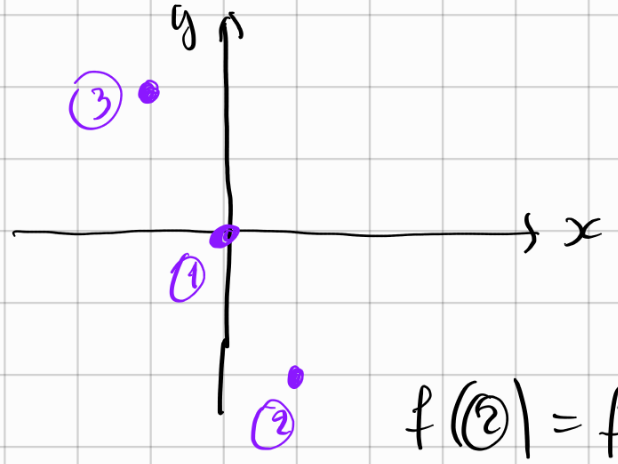
$$\begin{cases} y = -2x \\ x\underbrace{(1 - 32x^2)} = 0 \end{cases}$$

① $\begin{cases} x = 0 \\ y = 0 \end{cases}$

②③ $\begin{cases} x^2 = \frac{1}{32} \end{cases}$

② $\begin{cases} x = \frac{1}{4\sqrt{2}} \\ y = -\frac{1}{2\sqrt{2}} \end{cases}$

③ $\begin{cases} x = -\frac{1}{4\sqrt{2}} \\ y = \frac{1}{2\sqrt{2}} \end{cases}$



$$f(①) = f(0,0) = 0$$

$$f(②) = f\left(\frac{1}{4\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right) = -\frac{1}{4\sqrt{2}} \frac{1}{2\sqrt{2}} + \frac{1}{32} + \frac{1}{64}$$

$$= -\frac{1}{16} + \frac{1}{32} + \frac{1}{64} = \frac{-4+2+1}{64} = -\frac{1}{64}$$

① non era minimo assoluto.

$$f(③) = f\left(-\frac{1}{4\sqrt{2}}, \frac{1}{2\sqrt{2}}\right) = \dots = -\frac{1}{64}$$

Se c'è un minimo deve essere uno tra ② e ③ quindi entrambi sono minimi assoluti.

Condizione al II ordine $\quad f \in C^2$

Se x_0 è un minimo (massimo) locale per f allora la matrice $D^2 f(x_0)$ ha tutti gli autovalori ≥ 0. ($\leq$)

ES $\quad f = xy + x^2 + y^4 \qquad \nabla f = \begin{pmatrix} y + 2x \\ x + 4y^3 \end{pmatrix}$

$$D^2 f = \begin{pmatrix} 2 & 1 \\ 1 & 12y^2 \end{pmatrix}$$

$\det D^2 f = \lambda_1 \cdot \lambda_2$

$\operatorname{tr} D^2 f = \lambda_1 + \lambda_2$

$\det D^2 f = 24y^2 - 1$

$\operatorname{tr} D^2 f = 2 + 12y^2$

in 1 $x = 0$, $y = 0$ $\quad \det = -1$, $\operatorname{tr} = 2$ $\quad \lambda_1 < 0$, $\lambda_2 > 0$

in 2 e 3

$y^2 = \frac{1}{8}$

$$D^2 f = \begin{pmatrix} 2 & 1 \\ 1 & \frac{3}{2} \end{pmatrix}$$

$$\begin{cases} \det = 3 - 1 = 2 \\ \operatorname{tr} = 2 + \frac{3}{2} = \frac{7}{2} \end{cases}$$

$\lambda_1 \lambda_2 > 0$

$\lambda_1 + \lambda_2 > 0$

$\begin{cases} \lambda_1 > 0 \\ \lambda_2 > 0 \end{cases} \Rightarrow$ minimo locale.