LEZIONE 67

Integrali impropri o generalizzati

$f:(a,b] \to \mathbb{R}$

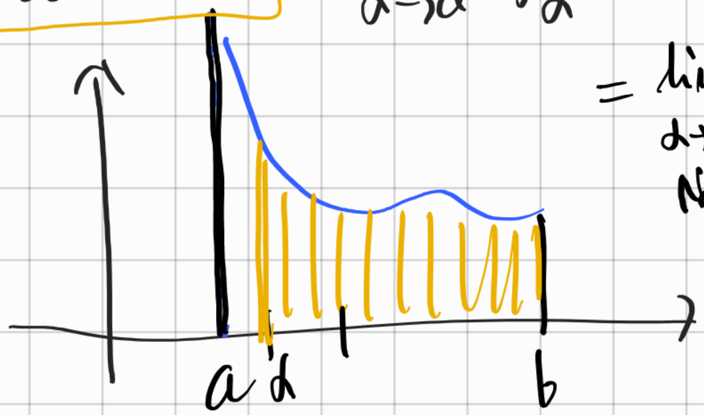
improprio

proprio la definizione di Riemann

$$\int_a^b f(x)\,dx := \lim_{\alpha\to a^+}\int_\alpha^b f(x)\,dx$$

$F'=f$

$$= \lim_{\alpha\to a^+}\left[F(x)\right]_\alpha^b \overset{\text{Notazione}}{=:} \left[F(x)\right]_a^b$$

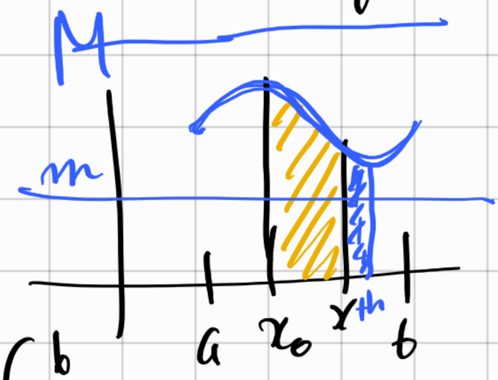


Oss Se $f:[a,b]\to\mathbb{R}$ è limitata e R-integrabile

$$F(x) = \int_{x_0}^{x} f(t)\,dt$$

F è continua.

In particolare $$\lim_{\alpha\to a^+}\int_\alpha^b f(x)\,dx = \int_a^b f(x)\,dx$$



dim

$$\lim_{h\to 0}\left[F(x+h)-F(x)\right] = \lim_{h\to 0}\int_x^{x+h} f(t)\,dt = 0$$

f è limitata $m \le f \le M$ $\quad h>0$

$$0 \xleftarrow[h\to 0]{} m\cdot h \le \int_x^{x+h} f(t)\,dt \le M\cdot h \to 0 \text{ per } h\to 0 \qquad \square$$

$$f:[a,b)\to\mathbb{R} \qquad \int_a^b f = \lim_{\beta\to b^-}\int_a^\beta f$$

$$f:(a,b)\to\mathbb{R}$$ scelgo $c\in(a,b)=(a,c]\cup[c,b)$

$$\int_a^b f = \int_a^c f + \int_c^b f . \qquad =[F]_a^b$$

se ha senso.

se F è una primitiva

Se I è un intervallo di estremi $a=\inf I$, $b=\sup I$

se esistono $\{x_0,\dots,x_n\}\subseteq I$ tali che

punti cattivi

$f: I\setminus\{x_0,\dots,x_n\}\to\mathbb{R}$ è localmente R-integrabile

definiamo $$\int_a^b f = \int_a^{x_0} f + \int_{x_0}^{x_1} f + \dots + \int_{x_n}^b f$$

se questa somma è ben definita.

Es $$\int_{-\infty}^{+\infty}\frac{1}{x}\,dx$$

$f(x)=\frac{1}{x}$

$f:(-\infty,0)\cup(0,+\infty)\to\mathbb{R}$

cattivi

$$\| $$

$$\int_{-\infty}^{0}\frac{1}{x}\,dx + \int_{0}^{+\infty}\frac{1}{x}\,dx$$

$$\| \qquad \qquad \|$$

$$\int_{-\infty}^{-1}\frac{1}{x}\,dx + \int_{-1}^{0}\frac{1}{x}\,dx + \int_{0}^{1}\frac{1}{x}\,dx + \int_{1}^{+\infty}\frac{1}{x}\,dx$$

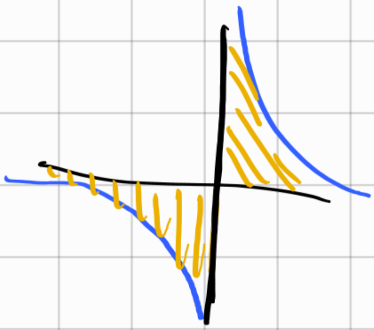
$$\underbrace{\left[\ln(-x)\right]_{-\infty}^{-1} + \left[\ln(-x)\right]_{-1}^{0}}_{=} + \left[\ln x\right]_0^1 + \left[\ln x\right]_1^{+\infty}$$

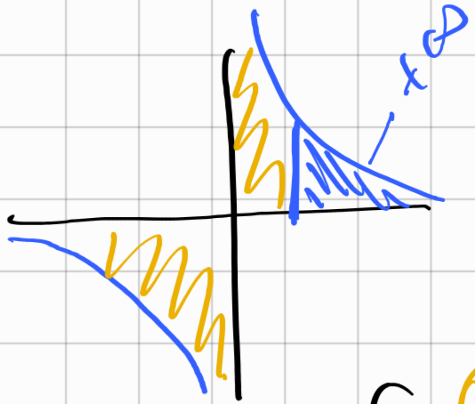
$$= \left[\ln(-x)\right]_{-\infty}^{0} + \left[\ln x\right]_0^{+\infty}$$

$$= (-\infty) - (+\infty) + (+\infty) - (-\infty)$$

$$= (-\infty) + (+\infty) \quad \text{non ha senso}$$

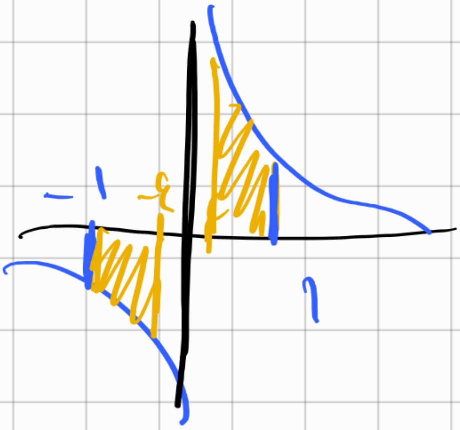
$\nexists$.





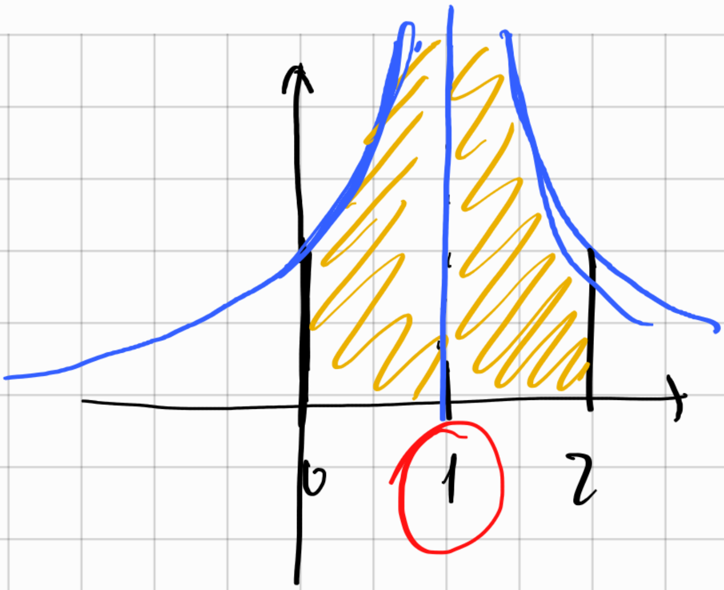
Oss (MA)

$$\lim_{\varepsilon \to 0} \underbrace{\left[\int_{-1}^{-\varepsilon} \frac{1}{x}\,dx + \int_{\varepsilon}^{1} \frac{1}{x}\,dx \right]}_{\overset{\shortparallel}{0}} = 0$$



NON È LA NOSTRA DEFINIZIONE

Es $\displaystyle\int_0^2 \frac{1}{\sqrt{|x-1|}}\,dx$



$$\int \frac{1}{\sqrt{x-1}}\,dx = \int (x-1)^{-\frac{1}{2}}\,dx$$
$$= 2\sqrt{x-1}$$

$$\int \frac{1}{\sqrt{1-x}}\,dx = \int (1-x)^{-\frac{1}{2}}\,dx = -2(1-x)^{\frac{1}{2}}$$

$y = 1-x$

$dy = -dx$

$$\int \frac{1}{\sqrt{|x-1|}}\,dx = \left\{\begin{array}{ll} 2\sqrt{x-1} & \text{se } x > 1 \\ -2\sqrt{1-x} & \text{se } x < 1 \end{array}\right\} = F(x)$$

$$\int_0^2 \frac{1}{\sqrt{|x-1|}}\,dx \overset{\triangle!}{=} \Big[F(x)\Big]_0^2 = \boxed{2} - \boxed{(-2)} = 4$$

NON È GIUSTIFICATO

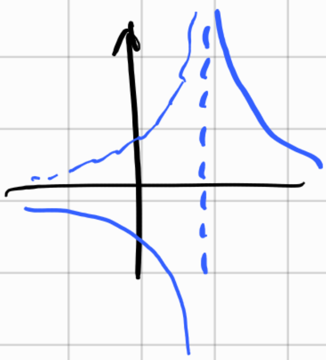
$$\int_0^2 \frac{1}{\sqrt{|x-1|}}\,dx = \int_0^1 \frac{1}{\sqrt{|x-1|}}\,dx + \int_1^2 \frac{1}{\sqrt{|x-1|}}$$

$$= \left[-2\sqrt{1-x}\right]_0^1 + \left[2\sqrt{x-1}\right]_1^2$$

$$= 0 - \boxed{(-2)} + \boxed{2} - 0 = 4$$

<u>Nota</u> $$\int_0^2 \frac{1}{x-1}\,dx \overset{\triangle}{=} \left[\ln|x-1|\right]_0^2 = \ln 1 - \ln 1 = 0 \quad \text{NO}$$

<u>Es</u> $$\int_0^2 \frac{1}{x-1}\,dx = \int_0^1 \frac{1}{x-1}\,dx + \int_1^2 \frac{1}{x-1}\,dx$$



$$= \left[\ln|x-1|\right]_0^1 + \left[\ln|x-1|\right]_1^2$$

$$= (-\infty) - \cancel{(\ln 1)} + (\ln 1) - (-\infty)$$

$$= \nexists$$ non è integrabile.

<u>Es</u> $$\int_0^2 \frac{1}{|x-1|}\,dx = +\infty$$

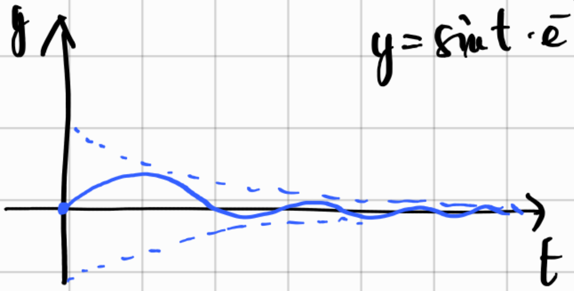
Es $\displaystyle\int_0^{+\infty}\left[\int_0^{+\infty} \sin t\cdot e^{-tx}\,dt\right]dx =$

$$\boxed{= \int_0^{+\infty} I(x)\,dx}$$

$t \geq 0$

$x \geq 0$

dove $\displaystyle I(x) = \int_0^{+\infty} \sin t \cdot e^{-tx}\cdot dt$



$y = \sin t \cdot e^{-tx}$

$x > 0$

$$I(x) = \int_0^{+\infty} \underset{d}{\sin t}\cdot \underset{\int}{e^{-tx}}\,dt = \left[\sin t \cdot \frac{e^{-tx}}{-x}\right]_{t=0}^{+\infty} - \int_0^{+\infty} \underset{\uparrow}{\cos t} \cdot \frac{e^{-tx}}{-x}\,dt$$

$$= 0 - 0 + \frac{1}{x}\int_0^{+\infty} \underset{d}{\cos t}\cdot \underset{\int}{e^{-tx}}\,dt$$

$$\left[\lim_{t\to\infty} \underset{\text{limitata}}{\sin t}\cdot \underset{\to 0}{\frac{e^{-tx}}{-x}} = 0\right]$$

$$= \frac{1}{x}\left[\left[\cos t \cdot \frac{e^{-tx}}{-x}\right]_{t=0}^{+\infty} - \int_0^{+\infty} (-\sin t)\cdot \frac{e^{-tx}}{-x}\,dt\right]$$

$$= \frac{1}{x}\left[0 - 1\cdot\frac{1}{-x} - \frac{1}{x}\int_0^{+\infty} \sin t \cdot e^{-tx}\,dt\right]$$

$$= \frac{1}{x^2} - \frac{1}{x^2}\, I(x)$$

$$x^2 I(x) = 1 - I(x)$$

$$(x^2+1) I(x) = 1$$

$$I(x) = \frac{1}{1+x^2}$$

$$\int_0^{+\infty} I(x)\,dx = \int_0^{+\infty} \frac{1}{1+x^2}\,dx = \left[\operatorname{arctg} x\right]_0^{+\infty}$$

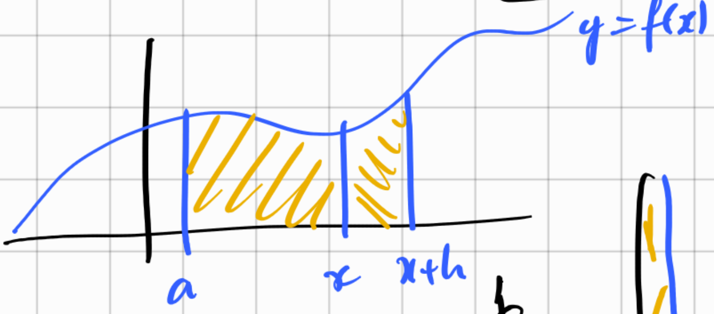
$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

<u>Oss</u> Se $f \geq 0$ $\int_a^b f(x)\,dx$ esiste

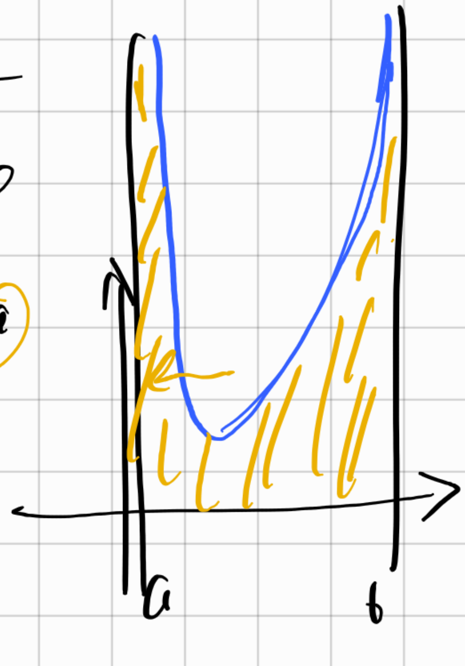
in questo $F(x) = \int_a^x f(t)\,dt$ se $f \geq 0 \Rightarrow F$ crescente.

$$F(x+h) = F(x) + \underbrace{\int_x^{x+h} \boxed{f(t)}\,dt}_{\geq 0} \geq F(x)$$

<u>se $h \geq 0$</u>

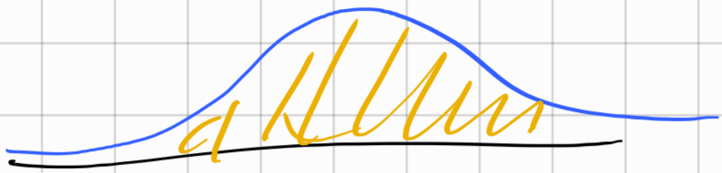


$$\int_a^b f(x)\,dx = \left[F\right]_a^b$$



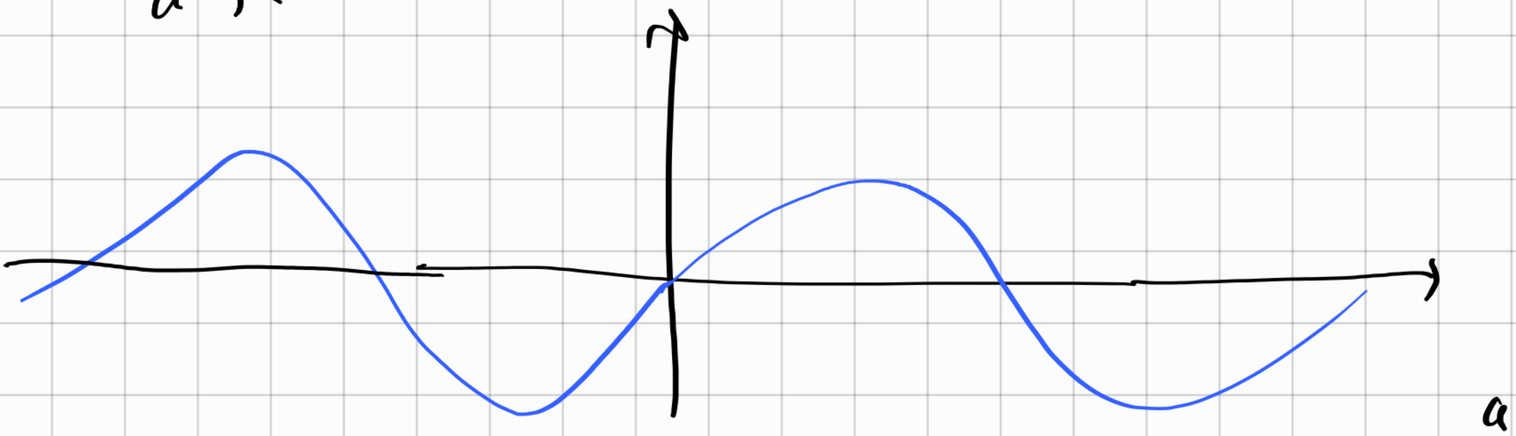
$\square$

ES $\int_{-\infty}^{+\infty} e^{-x^2}\,dx$ esiste.



ES $\int_{-\infty}^{+\infty} \sin x\,dx = \nexists \qquad = \left[-\cos x\right]_{-\infty}^{+\infty}$

$$\lim_{a\to+\infty}\int_{-a}^{a}\sin x\,dx = 0$$



$$\lim_{a\to+\infty}\int_{1-a}^{1+a}\sin x\,dx$$

$$\int_{-\infty}^{+\infty}\sin x\,dx = \int_{-\infty-1}^{+\infty-1}\sin(y-1)\,dy$$

$y = x+1$ ...