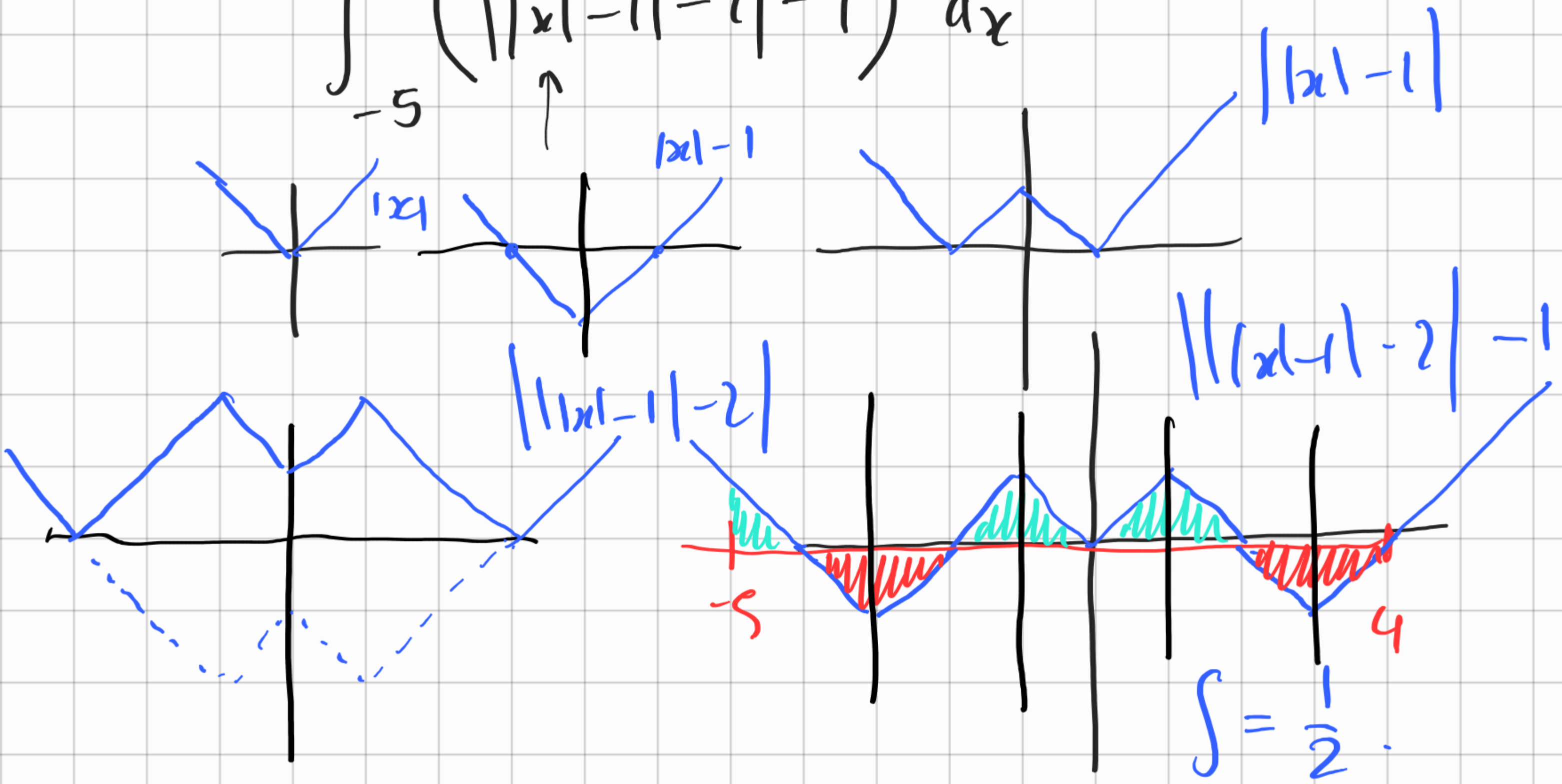
ANALISI MATEMATICA B
LEZIONE 63 - 7.3.2022

Test settimanale   Es 4

$$\int_{-5}^{4}\left(\left|\left||x|-1\right|-2\right|-1\right)dx$$



$\int = \frac{1}{2}$.

Diversi concetti:

$\boxed{\int_a^b f(x)dx \in \mathbb{R}}$ integrale

- $x \longmapsto \int_{x_0}^{x} f(t)\,dt$ funzione integrale
- $\int f(x)dx$ primitiva (integrale indefinito)

Th (Torricelli-Barrow: fondamentale del calcolo)

la funzione integrale è una primitiva.

Integrazione per sostituzione:

(i) $$\int f(g(x))\,\boxed{g'(x)}\,dx = \left[\int f(y)\,dy\right]_{y=g(x)}$$

sostituzione diretta

$y = g(x)$
$dy = g'(x)\,dx$

$F' = f$ $$\left(F(g(x))\right)' = F'(g(x))\cdot g'(x)$$
$$= f(g(x))\cdot g'(x).$$

(ii) $$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(y)\,dy$$

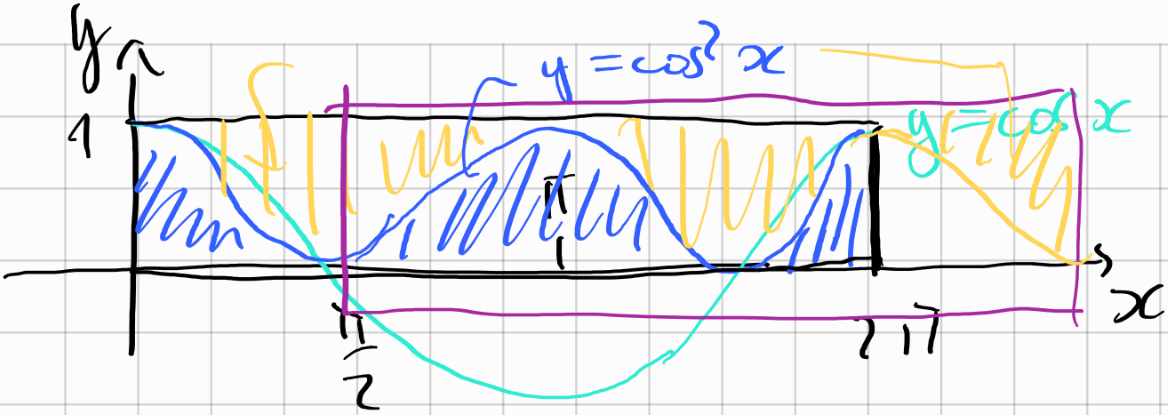
$$\left[F(g(x))\right]_a^b = \left[F(y)\right]_{g(a)}^{g(b)}$$

<u>Es</u> $$\int_0^{2\pi} \cos^2 x\,dx = \int_0^{2\pi} \boxed{\frac{1+\cos(2x)}{2}}\,dx$$

$\cos^2 x$ ha periodo π

$y = 2x$
$dy = 2\,dx$

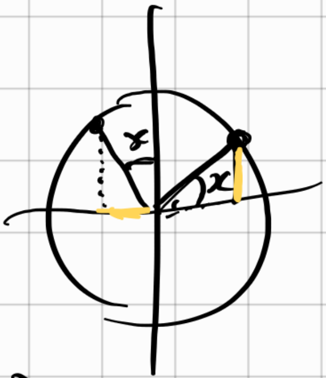
$$= \int_0^{2\pi} \frac{1}{2} + \frac{1}{4}\int_0^{2\pi} \cos(2x)\,2\,dx$$
$$= \pi + \frac{1}{4}\int_0^{4\pi} \cos(y)\,dy$$
$$= \pi + \frac{1}{4}\left[\sin(y)\right]_0^{4\pi}$$
$$= \pi + \frac{1}{4}\left[\cancel{\sin(4\pi)} - \cancel{\sin(0)}\right] = \pi.$$



$1 - \cos^2 x = \sin^2 x$

$$\int_0^{2\pi} \cos^2 x \, dx = \int_0^{2\pi} \sin^2 x \, dx$$

$$\begin{cases} x = \frac{\pi}{2} + y \\ dx = dy \end{cases}$$



$$-\sin y = \cos\left(\frac{\pi}{2} + y\right)$$

$$\int_0^{2\pi} \cos^2\left(\frac{\pi}{2} + y\right) dy = \int_{\frac{\pi}{2}}^{2\pi + \pi/2} \left(-\sin y\right)^2 dy =$$

$$= \int_{\frac{\pi}{2}}^{2\pi} \sin^2 y \, dy + \int_{2\pi}^{2\pi + \pi/2} \sin^2 y \, dy$$

$$\sin(z) = \sin(z + 2\pi) = \sin(y)$$

$$\begin{cases} z = y - 2\pi \\ dz = dy \end{cases}$$

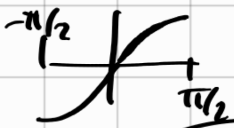
$$= \int_{\frac{\pi}{2}}^{2\pi} \sin^2 y \, dy + \int_0^{\pi/2} \sin^2 z \, dz$$

$$= \int_0^{2\pi} \sin^2 y \, dy .$$

(iii) Sostituzione inversa

$$\boxed{\int f(x)\, dx = \left[\int f(g(t))\, g'(t)\, dt \right]_{t = g^{-1}(x)}}$$

$$\begin{cases} x = g(t) \\ dx = g'(t)\, dt \\ t = g^{-1}(x) \end{cases}$$



Es $\displaystyle\int \sqrt{1-x^2}\, dx$ $\qquad \begin{cases} x = \sin t \\ dx = \cos t \, dt \\ t = \arcsin x \end{cases}$ $\quad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$= \int \sqrt{1 - \sin^2 t} \cdot \cos t \, dt$$

$$\cos t = \sqrt{1 - \sin^2 t} = \sqrt{1-x^2}$$

$$= \int |\cos t| \cdot \cos t \, dt = \int \cos^2 t \, dt$$

$$= \frac{t + \sin t \cdot \cos t}{2} = \boxed{\frac{\arcsin x + x \cdot \sqrt{1-x^2}}{2}}$$

$$(1-x^2)^{\frac{1}{2}}$$

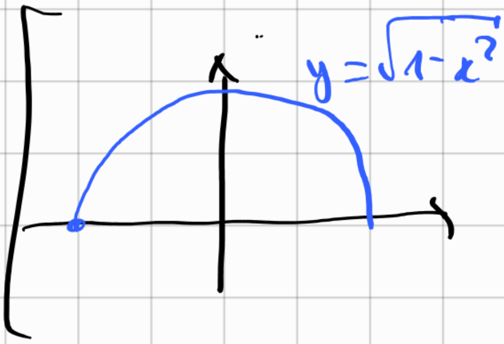
Verifica

$$\left(\frac{\arcsin x + x\sqrt{1-x^2}}{2} \right)' = \frac{1}{2}\left[\frac{1}{\sqrt{1-x^2}} + \sqrt{1-x^2} + x \frac{1}{2} \cdot \frac{-2x}{\sqrt{1-x^2}} \right]$$

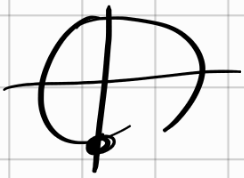
$$= \frac{1}{2} \, \frac{1 + (1-x^2) - x^2}{\sqrt{1-x^2}}$$

$$= \frac{1-x^2}{\sqrt{1-x^2}} = \sqrt{1-x^2} \qquad \square$$

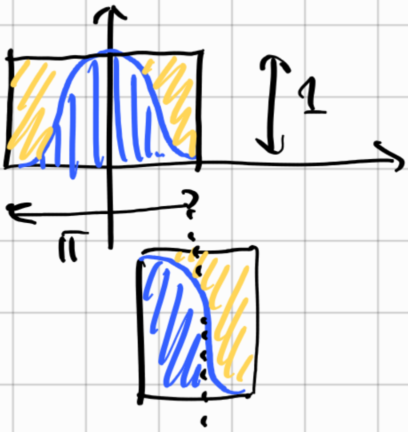
ES $\displaystyle\int_{-1}^{1}\sqrt{1-x^2}\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{1-\sin^2 t}\cdot\cos t\,dt = (*)$



$$\begin{cases} x = \sin t \\ dx = \cos t\,dt \\ t = \arcsin x \end{cases} \qquad t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$



$$(*) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\,dt = \frac{\pi}{2}$$



$$\cos^2 x = \sin^2\left(\frac{\pi}{2}-x\right)$$
$$= 1-\sin^2 x$$

dim $\displaystyle\int_a^b f(x)\,dx = \left[\int_{g^{-1}(a)}^{g^{-1}(b)} f(g(t))\cdot g'(t)\,dt\right]_{t=g^{-1}(x)}$

$$\boxed{H(t)\in\int f(g(t))\,g'(t)\,dt}$$

$F(x) = H(g^{-1}(x))$ $\qquad F \overset{?}{\in} \int f.$

$$F'(x) = H'(g^{-1}(x))\cdot\frac{1}{g'(g^{-1}(x))} =$$

$$= f\left(g\left(g^{-1}(x)\right)\right)\cdot \cancel{g'\left(g^{-1}(x)\right)}\cdot \frac{1}{\cancel{g'\left(g^{-1}(x)\right)}}$$

$$= f(x).$$

<u>Integrazione per parti</u> $\qquad (F\cdot g)' = \underbrace{F'g} + F\cdot g'$

$f = F'$ $\qquad F'\cdot g = \underbrace{(F\cdot g)'} - F\cdot g'$

$$\underbrace{\int f\cdot g} = \int F'\cdot g = \int\left[(F\cdot g)' - F\cdot g'\right]$$

$$= \underbrace{F\cdot g} - \int F\cdot g'$$

$F = \int f$

$$\left[\boxed{\int f\cdot g = F\cdot g - \int F\cdot g'}\right]_a^b \quad \text{dove } F \in \int f.$$

$$\int_a^b f\cdot g = \left[F\cdot g\right]_a^b - \int_a^b F\cdot g'$$

$$\underbrace{\int_a^b f(x)\, g(x)\, dx = \left[F(x)\cdot g(x)\right]_a^b - \int_a^b F(x)\, g'(x)\, dx}$$

<u>Esempio</u> $\quad \int x\cdot \cos x\, dx = x\cdot \sin x - \int 1\cdot \sin x\, dx$

$g(x) = x \qquad F(x) = \sin x$

$f(x) = \cos x$

$$= x\cdot \sin x + \cos x$$

Verifica: $D\left(x\sin x + \cos x\right) = 1\cdot\cancel{\sin x} + x\cdot\cos x - \cancel{\sin x} = x\cdot\cos x.$

Per casa $\int x^2\cdot\sin x\,dx$

Per casa $\int \left(x^2-3x\right)\cdot e^{2x}\,dx$ $\int x\,e^{x^2}\,dx$

$\int x\,e^{x^2-x}\,dx$

per parti

Es $\int e^x\cdot\cos x\,dx = e^x\sin x - \int e^x\sin x\,dx$

$= e^x\sin x - \left[e^x\cdot(-\cos x) - \int e^x(-\cos x)\,dx\right]$

$= e^x\sin x + e^x\cos x - \int e^x\cos x\,dx$

$2\int e^x\cos x\,dx = e^x\sin x + e^x\cos x$

$\int e^x\cos x\,dx = \frac{1}{2}e^x\left(\sin x + \cos x\right).$