ANALISI MATEMATICA B
LEZIONE 56 - 16.2.2022

FUNZIONI ANALITICHE

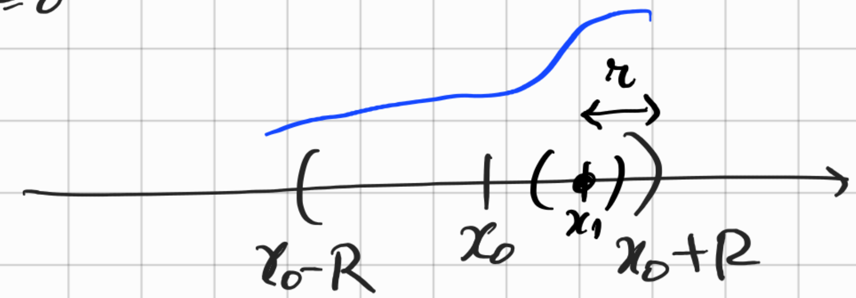
$f: A \subseteq \mathbb{R} \to \mathbb{R}$ si dice <u>analitica</u> se

$\forall x_0 \in A \quad \exists \rho > 0 \quad \exists a_k : |x-x_0|<\rho$

$$f(x)=\sum_{k=0}^{+\infty} a_k (x-x_0)^k .$$

Se

$$f(x)=\sum_{k=0}^{+\infty} a_k (x-x_0)^k \quad \text{per } |x-x_0|<R \ (\leq +\infty)$$



Se $|x_1-x_0|<R$ e $|x-x_1|<r=R-|x_1-x_0|$

allora $$f(x)=\sum_{k=0}^{+\infty} b_k (x-x_1)^k$$

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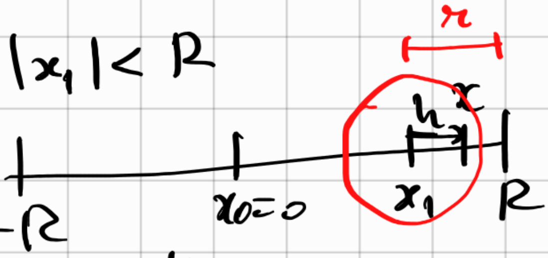
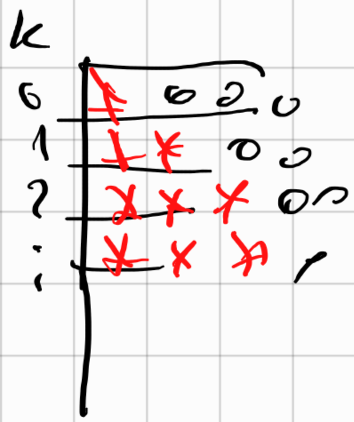
dim $x_0=0$ per comodità. $|x_1|<R$

$|x-x_1|<r=R-|x_1|$. $h=x-x_1$



$$f(x)=\sum_{k=0}^{+\infty} a_k x^k = \sum_{k=0}^{+\infty} a_k (x_1+h)^k$$



$$= \sum_{k=0}^{+\infty} a_k \boxed{\sum_{j=0}^{k} \binom{k}{j} x_1^{k-j} h^j}$$

$$= \sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} c_{k,j} \qquad c_{k,j} = \begin{cases} a_k \binom{k}{j} x_1^{k-j} h^j & \text{se } j \le k \\ 0 & \text{se } j > k \end{cases}$$

$$\overset{?}{=} \sum_{j=0}^{+\infty} \sum_{k=0}^{+\infty} c_{k,j} = \sum_{j=0}^{+\infty} \underbrace{\left[\sum_{k=j}^{+\infty} a_k \binom{k}{j} x_1^{k-j} \right]}_{b_j} h^j$$

Teor. Se $\boxed{\sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} |c_{k,j}| < +\infty}$ ‖

Allora
$$\sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} c_{k,j} = \sum_{n=0}^{+\infty} \boxed{\sum_{k=0}^{n} c_{k,n-k}}$$

ma anche
$$= \sum_{j=0}^{+\infty} \sum_{k=0}^{+\infty} c_{k,j}.$$

$$\sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} |c_{k,j}| = \sum_{k=0}^{+\infty} |a_k| \sum_{j=0}^{k} \binom{k}{j} |x_1|^{k-j} |h|^j$$

$$= \sum_{k=0}^{+\infty} |a_k| \left(|x_1| + |h| \right)^k$$

converge perché $\sum a_k x^k$ converge assolutamente se $|x| < R$

$$|x_1| + |h| < R \qquad \square$$

Se

$(\forall)$ $$f(x)=\sum_{k=0}^{+\infty} a_k (x-x_0)^k \qquad R = \text{raggio di conv.}$$

se $|x|<R$

Allora $$f'(x)=\sum_{k=1}^{+\infty} k\cdot a_k (x-x_0)^{k-1} = \sum_{k=0}^{+\infty} (k+1)a_{k+1}(x-x_0)^k$$

Conseguenza: f analitica $\Rightarrow f\in C^\infty$

$$f''(x)=\sum_{k=2}^{+\infty} k(k-1)a_k (x-x_0)^{k-2}$$

$\vdots$

$$f^{(n)}(x)=\sum_{k=n}^{+\infty} \overbrace{k(k-1)\dots(k-n+1)}^{n \text{ fattori}} a_k (x-x_0)^{k-n}$$

$$f^{(n)}(x_0) = n!\cdot a_k \qquad \boxed{a_k=\frac{f^{(n)}(x_0)}{n!}}$$

quindi $\sum_{k=0}^{n} a_k(x-x_0)^k$ è il polinomio di Taylor di f centrato in x_0 di ordine n.

[Abbiamo già visto $f(x)=\begin{cases} e^{-\frac{1}{x^2}} & \text{se } x\neq 0 \\ 0 & \text{se } x=0\end{cases}$

$\forall n\in\mathbb{N}$: $f^{(n)}(0)=0$ $\qquad \boxed{a_k}=0$

è C^∞ ma non è analitica $\qquad \sum a_k x^k = 0 \neq f(x)$ se $x\neq 0$]

dim $(\forall)$ $\qquad x_0=0 \qquad f(x)=\sum_{k=0}^{+\infty} a_k x^k \quad R$

$|x|<R \quad |h|<R-|x|$

$$\frac{f(x+h)-f(x)}{h} = \frac{1}{h}\left[\sum_{k=0}^{+\infty} a_k (x+h)^k - \sum_{k=0}^{+\infty} a_k x^k\right]$$

$$= \frac{1}{h} \sum_{k=0}^{+\infty} a_k \left[(x+h)^k - x^k\right]$$

$$= \frac{1}{h} \sum_{k=0}^{+\infty} a_k \left[\sum_{j=0}^{k} \binom{k}{j} x^{k-j} h^j - x^k\right]$$

$$= \sum_{k=0}^{+\infty} a_k \sum_{j=1}^{k} \binom{k}{j} x^{k-j} h^{j-1}$$

$j \leq k$

$$\overset{?}{=} \sum_{j=1}^{+\infty} \left[\sum_{k=j}^{+\infty} a_k \binom{k}{j} x^{k-j}\right] h^{j-1}$$

$$= \sum_{j=0}^{+\infty} b_j h^j \longrightarrow b_0 = \sum_{k=1}^{+\infty} a_k \binom{k}{1} x^{k-1} = \sum_{k=1}^{+\infty} k a_k x^{k-1} \quad \square$$

$$c_{k,j} = \begin{cases} a_k \binom{k}{j} x^{k-j} h^{j-1} & j=1 \dots k \\ 0 & \text{se } j=0 \\ 0 & \text{se } j>k \end{cases}$$

$$\sum_{k} \sum_{j} |c_{k,j}| \overset{?}{<} +\infty$$

$$\sum_{k=0}^{+\infty} \sum_{j=1}^{k} |a_k| \underbrace{\binom{k}{j} |x|^{k-j} |h|^{j}}$$

$$= \sum_{k=0}^{+\infty} |a_k| \left((|x|+|h|)^k - |x|^k \right)$$

$$= \sum_{k=0}^{+\infty} |a_k| (|x|+|h|)^k - \sum_{k=0}^{+\infty} |a_k| |x|^k \in \mathbb{R}$$

$|x|+|h| < R$ $\qquad$ $|x| < R$ $\qquad \square$

Teorema: Se $|x|<1$: $(1+x)^\alpha = \sum_{k=0}^{+\infty} \binom{\alpha}{k} x^k \qquad \alpha \in \mathbb{R}$

quindi $f(x) = (1+x)^\alpha$ è analitica su $(-1, 1)$.

dim 1. $R=1$ la serie converge se $|x|<1$.

$$\frac{1}{R} = \lim_{k \to +\infty} \frac{\left|\binom{\alpha}{k+1}\right|}{\left|\binom{\alpha}{k}\right|}$$

$$\frac{\left|\binom{\alpha}{k+1}\right|}{\left|\binom{\alpha}{k}\right|} = \frac{\left| \frac{\overbrace{\cancel{\alpha(\alpha-1)\cdots}(\alpha-k)}^{k+1}}{(k+1)\cancel{!}} \right|}{\left| \frac{\cancel{\alpha(\alpha-1)\cdots(\alpha-k+1)}}{\cancel{k!}} \right|}$$

$$= \frac{|\alpha-k|}{k+1} \to 1 \quad \text{per } k \to +\infty$$

$R = 1.$

2. Posto $f(x) = \sum_{k=0}^{+\infty} \binom{\alpha}{k} x^k$ per $|x| < 1$.

voglio mostrare che $f(x) = (1+x)^\alpha$

ovvero $\dfrac{f(x)}{(1+x)^\alpha} = 1$ $\forall x \in (-1,1)$

basta mostrare che (a) $\dfrac{f(0)}{(1+0)^\alpha} = 1$

e che (b) $\left(\dfrac{f(x)}{(1+x)^\alpha} \right)' = 0$ $\forall x.$

(a) $f(0) = \binom{\alpha}{0} = 1$

(b) $\underline{\left(f(x)(1+x)^{-\alpha} \right)'} = \dfrac{f'(x)}{(1+x)^\alpha} - \dfrac{\alpha f(x)}{(1+x)^{\alpha+1}}$

$= \dfrac{f'(x)(1+x) - \alpha f(x)}{(1+x)^{\alpha+1}} \overset{?}{=} 0$ $\forall x \in (-1,1)$

Basta mostrare che $\underline{f'(x) \cdot (1+x) = \alpha f(x)}$ $\forall |x| < 1$

$f(x) = \sum_{k=0}^{+\infty} \binom{\alpha}{k} x^k, \quad f'(x) = \sum_{k=1}^{+\infty} k \binom{\alpha}{k} x^{k-1}$

$$f'(x)\cdot(1+x) = \sum_{k=1}^{+\infty} k\binom{\alpha}{k}x^{k-1} + \sum_{k=1}^{+\infty} k\binom{\alpha}{k}x^{k}$$

$$= \sum_{k=0}^{+\infty}(k+1)\binom{\alpha}{k+1}x^k + \sum_{k=0}^{+\infty} k\binom{\alpha}{k}x^k$$

$$= \sum_{k=0}^{+\infty}\left[(k+1)\binom{\alpha}{k+1} + k\binom{\alpha}{k}\right]\cdot x^k$$

$$\stackrel{?}{=} \sum_{k=0}^{+\infty} \alpha\cdot\binom{\alpha}{k}x^k$$

$$(k+1)\binom{\alpha}{k+1} + k\binom{\alpha}{k} = (k+1)\frac{\alpha(\alpha-1)\dots(\alpha-k)}{(k+1)!} + k\,\frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!}$$

$$= \frac{\alpha(\alpha-1)\dots(\alpha-k) + k\cdot\alpha(\alpha-1)\dots(\alpha-k+1)}{k!}$$

$$= \boxed{\frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!}}\left[(\alpha-\cancel{k})+\cancel{k}\right] \stackrel{!!}{=} \binom{\alpha}{k}\cdot\alpha$$

□

<u>Corollario</u> $f(x)=x^\alpha$ è analitica su $(0,+\infty)$

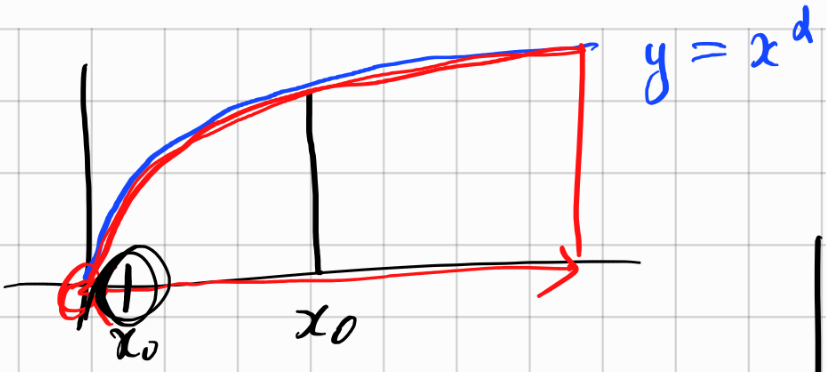
Sia $x_0>0$

$$f(x)=x^\alpha = (x_0+x-x_0)^\alpha = x_0^\alpha\underbrace{\Big(1+\overbrace{\frac{x-x_0}{x_0}}^{t}\Big)^\alpha}$$

$$= x_0^\alpha\sum_{k=0}^{+\infty}\binom{\alpha}{k}\left(\frac{x-x_0}{x_0}\right)^k \qquad |x-x_0|<|x_0|$$

$$= x_0^\alpha\sum_{k=0}^{+\infty}\frac{\binom{\alpha}{k}}{x_0^k}(x-x_0)^k$$

□



<u>OSS</u> se $f'(x)$ è analitica anche $f(x)$ analitica.

$$\left(\ln(1+x)\right)' = (1+x)^{-1} \leftarrow \text{è analitica}$$

$$\left(\operatorname{arctg} x\right)' = (1+x^2)^{-1} \leftarrow \text{è analitica}$$

$$\left(\arcsin x\right)' = (1-x^2)^{-\frac{1}{2}} \leftarrow \quad ''$$

$$\left(\arccos x\right)' = -(1-x^2)^{-\frac{1}{2}} \quad ''$$

$$\arcsin x = x + \frac{x^3}{6} + \dots + \frac{(2n-1)!!}{(2n)!!}\,\frac{1}{(2n+1)}\,x^{2n+1} + \dots\dots$$

$$\frac{\pi}{6} = \arcsin\frac{1}{2} = \frac{1}{2} + \frac{1}{48} + \dots$$

$$\pi = \overbrace{3 + \frac{1}{8} + \frac{9}{640} + \frac{45}{21504}}^{3.141} + \left(\dots\dots\right) < 0.0018$$