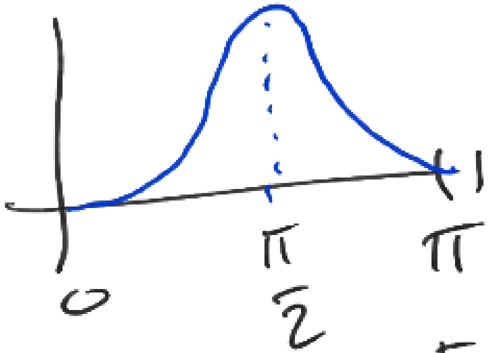
Irrazionalità di pi

$$\pi \notin \mathbb{Q}$$

$$I_n = \int_0^\pi x^n(\pi - x)^n \sin x \, dx$$



$$I_0 = \int_0^\pi \sin x \, dx = \left[-\cos x\right]_0^\pi = 2.$$

$$I_1 = \int_0^\pi x(\pi - x) \sin x \, dx$$

per parti

$$= \cancel{\left[x(\pi - x)(-\cos x)\right]_0^\pi} - \int_0^\pi (\pi - 2x)(-\cos x)\, dx$$

$$= \int_0^\pi \pi \cos x \, dx - 2\int_0^\pi x \cos x \, dx$$

$$= \pi \cancel{\left[\sin x\right]_0^\pi} - 2\left(\cancel{\left[x \sin x\right]_0^\pi} - \int_0^\pi \sin x \, dx\right)$$

$$= 2 \cdot 2 = 4.$$

$$I_n = \int_0^\pi x^n (\pi - x)^n \sin x \, dx \qquad n \geq 1$$

per parti

$$= \left[x^n (\pi - x)^n (-\cos x) \right]_0^\pi$$

$$- \int_0^\pi \left(n x^{n-1} (\pi - x)^n - x^n n (\pi - x)^{n-1} \right) \cdot (-\cos x) \, dx$$

$$= \int_0^\pi \left(n x^{n-1} (\pi - x)^n - n x^n (\pi - x)^{n-1} \right) \cos x \, dx$$

per parti

$n \geq 2$

$$= 0 - \int_0^\pi \left(n(n-1) x^{n-2} (\pi - x)^n - 2n^2 x^{n-1} (\pi - x)^{n-1} + n(n-1) x^n (\pi - x)^{n-2} \right) \sin x \, dx$$

$$= 2n^2 I_{n-1} - \int_0^\pi (n^2 - n) \left[(\pi - x)^2 + x^2 \right] x^{n-2} (\pi - x)^{n-2} \sin x \, dx$$

$$a^2 + b^2 = (a+b)^2 - 2ab$$

$$= 2n^2 I_{n-1} - (n^2 - n) \int_0^\pi \left[\pi^2 - 2x(\pi - x) \right] x^{n-2} (\pi - x)^{n-2} \sin x \, dx$$

$$I_n = 2n^2 I_{n-1} - (n^2 - n) \left[\pi^2 I_{n-2} - 2 I_{n-1} \right]$$

$$= \left[2n^2 + (n^2 - n) \cdot 2 \right] I_{n-1} - \pi^2 (n^2 - n) I_{n-2}$$

$$\boxed{\begin{cases} I_n = (4n^2 - 2n)\, I_{n-1} - \pi^2 (n^2 - n)\, I_{n-2} \\ I_0 = 2 \\ I_1 = 4 \end{cases}}$$

Supponiamo che $\pi \in \mathbb{Q}$ per assurdo.

$$\pi = \frac{p}{q}, \quad p, q \in \mathbb{N}.$$

$$\boxed{a_n = \frac{I_n \cdot q^{2n}}{n!}} \qquad I_n = \frac{n!\, a_n}{q^{2n}}$$

$$\begin{cases} a_0 = 2 \\ a_1 = 4 \cdot q^2 \\ a_n = \dfrac{q^{2n}}{n!} \left((4n^2 - 2n)\, I_{n-1} - \pi^2 (n^2 - n)\, I_{n-2} \right) \end{cases}$$

$$= \frac{q^{2n}}{n!} \left((4n^2 - 2n) \frac{(n-1)!\, a_{n-1}}{q^{2n-2}} - \frac{p^2}{q^2} (n^2 - n) \frac{(n-2)!\, a_{n-2}}{q^{2n-4}} \right)$$

$a_n \in \mathbb{Z}$? ok!

$$q^{2n} \left((4n-2) \cdot \overbrace{n \cdot (n-1)!}^{n!}\, a_{n-1} + \right.$$

$$= \frac{\cdot}{n!}\Big(\quad \frac{q^{2n-2}}{\quad} - p^2 \frac{\overbrace{n(n-1)(n-2)!}^{n!}}{q^{2n-2}} a_{n-2} \Big)$$

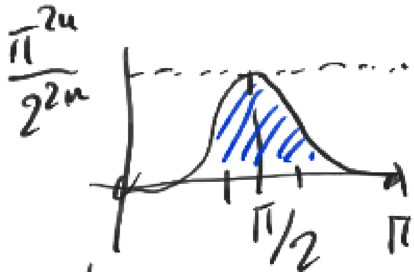
$$= q^2\big((4n-2)a_{n-1} - p^2 a_{n-2}\big) \in \mathbb{Z}$$

se $a_{n-1}, a_{n-2} \in \mathbb{Z}$.

Idea: mostrare che $a_n > 0$

e $a_n \to 0$.

$$a_n = \frac{q^{2n}}{n!}\int_0^{\pi} x^n(\pi - x)^n \sin x \, dx.$$



$a_n > 0$ in quanto la funzione integranda è > 0 in un intervallo.

$$a_n \le \frac{q^{2n}}{n!}\int_0^{\pi} \frac{\pi^{2n}}{2^{2n}} \, dx$$

$$= \frac{q^{2n}}{\quad} \cdot \frac{\pi^{2n+1}}{\quad} \longrightarrow 0$$

$n!$ 4

in questo

$n! >> c^n$.

$a_n \in \mathbb{Z}$, $a_n > 0 \Rightarrow a_n \geq 1$.

$a_n \to 0$ ASSURDO

$\square$