Fourier transform and the Sobolev space $H^1(\mathbb{R})$.

The Fourier transform for complex-valued functions in $\mathbb{R}$

For every function $\varphi \in C_c^\infty(\mathbb{R};\mathbb{C})$, we define its Fourier transform $\widehat{\varphi} : \mathbb{R} \to \mathbb{C}$ as

$$\mathcal{F}(\varphi) = \widehat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} \varphi(x)\, dx.$$

We recall the following properties of the Fourier transform $\mathcal{F}$:

Proposition 1. *For every $\varphi \in C_c^\infty(\mathbb{R},\mathbb{C})$, we have that*

$$\mathcal{F}(\varphi')(\xi) = i\xi\widehat{\varphi}(\xi).$$

Proposition 2. *For every $\varphi \in C_c^\infty(\mathbb{R},\mathbb{C})$, we have that*

$$\|\widehat{\varphi}\|_{L^2}^2 = \|\varphi\|_{L^2}^2.$$

Proposition 3. *The Fourier transform*

$$\mathcal{F} : C_c^\infty(\mathbb{R},\mathbb{C}) \to L^2(\mathbb{R},\mathbb{C}),$$

extends to a bounded linear functional

$$\mathcal{F} : L^2(\mathbb{R},\mathbb{C}) \to L^2(\mathbb{R},\mathbb{C}),$$

such that

$$\int_{\mathbb{R}} u(x)\overline{v(x)}\, dx = \int_{\mathbb{R}} \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\, d\xi \quad \textit{for all} \quad u, v \in L^2(\mathbb{R};\mathbb{C}),$$

and, in particular,

$$\|\mathcal{F}(u)\|_{L^2}^2 = \|u\|_{L^2}^2 \quad \textit{for all} \quad u \in L^2(\mathbb{R};\mathbb{C}).$$

Moreover, $\mathcal{F}$ is invertible and its inverse is given by

$$\mathcal{F}^{-1}(\psi)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} \psi(\xi)\, d\xi.$$

Fourier transform and (real-valued) Sobolev functions

Theorem 4. *Given a function $u \in L^2(\mathbb{R})$, the following are equivalent:*

(1) $u \in H^1(\mathbb{R})$;

(2) $|\xi||\widehat{u}|(\xi) \in L^2_\xi(\mathbb{R})$.

Moreover, for the weak derivative $u' \in L^2(\mathbb{R})$ we have the identity

$$\widehat{u'}(\xi) = i\xi\widehat{u}(\xi).$$

Proof. We first prove that (1) implies (2).
Let $\varphi_n \in C_c^\infty(\mathbb{R})$ be a sequence that converges strongly in $H^1(\mathbb{R})$ to u. First, we notice that the strong L^2 convergence of the functions and their derivatives implies that

$$\varphi_n \to u \qquad \text{and} \qquad \widehat{\varphi}_n \to \widehat{u}\,,$$
$$\varphi_n' \to u' \qquad \text{and} \qquad \widehat{\varphi_n'} \to \widehat{u'}\,,$$

strongly in L^2. Moreover, since

$$\|\widehat{\varphi_n'} - \widehat{\varphi_m'}\|_{L^2} = \big\||\xi|\big(\widehat{\varphi_n}(\xi) - \widehat{\varphi_m}(\xi)\big)\big\|_{L^2},$$

we get that the sequence $\xi\widehat{\varphi_n}(\xi)$ is a Cauchy sequence in $L^2(\mathbb{R};\mathbb{C})$. Let $v \in L^2(\mathbb{R};\mathbb{C})$ be the strong L^2 limit of $i\xi\widehat{\varphi_n}(\xi)$. Since the L^2 convergence implies the pointwise convergence along subsequences we obtain the following pointwise limits

$$\widehat{u}(\xi) = \lim_{n\to+\infty} \widehat{\varphi}_n(\xi) \qquad \text{and} \qquad v(\xi) = \lim_{n\to+\infty} i\xi\widehat{\varphi_n}(\xi).$$

Thus,

$$v(\xi) = i\xi\widehat{u}(\xi),$$

and so we get

$$i\xi\widehat{u}(\xi) = \lim_{n\to+\infty} i\xi\widehat{\varphi}_n(\xi) = \lim_{n\to+\infty} \widehat{\varphi_n'}(\xi) = \widehat{u'}(\xi).$$

We will next show that (2) implies (1). Thanks to the fact that the Fourier transform is an isometry, we can find a function $v \in L^2(\mathbb{R};\mathbb{C})$ such that

$$\widehat{v}(\xi) = i\xi\widehat{u}(\xi).$$

We will prove that v is the weak derivative of u. Consider a function $\varphi \in C_c^\infty(\mathbb{R})$. Then,

$$\begin{aligned}
\int_{\mathbb{R}} u(x)\varphi'(x)\,dx &= \int_{\mathbb{R}} \widehat{u}(\xi)\overline{\widehat{\varphi'}(\xi)}\,d\xi \\
&= \int_{\mathbb{R}} \widehat{u}(\xi)\overline{i\xi\widehat{\varphi}(\xi)}\,d\xi \\
&= -\int_{\mathbb{R}} i\xi\widehat{u}(\xi)\overline{\widehat{\varphi}(\xi)}\,d\xi \\
&= -\int_{\mathbb{R}} \widehat{v}(\xi)\overline{\widehat{\varphi}(\xi)}\,d\xi \\
&= -\int_{\mathbb{R}} v(x)\varphi(x)\,dx.
\end{aligned}$$

We now notice that $v(x)$ is a real valued function. Indeed, if we write $v = v_1 + iv_2$ for two real-valued $v_{1,2} \in L^2(\mathbb{R})$ we get that for all real-valued $\varphi \in C_c^\infty(\mathbb{R})$, we have:

$$\begin{aligned}
\int_{\mathbb{R}} u(x)\varphi'(x)\,dx &= -\int_{\mathbb{R}} v(x)\varphi(x)\,dx \\
&= -\int_{\mathbb{R}} v_1(x)\varphi(x)\,dx - i\int_{\mathbb{R}} v_2(x)\varphi(x)\,dx,
\end{aligned}$$

which implies that

$$\int_{\mathbb{R}} v_2(x)\varphi(x)\,dx = 0.$$

Since, φ is arbitrary, we get that $v_2 = 0$. This concludes the proof. $\square$