Exercices

Exercice 1. *Let $u \in W^{1,p}((a,b))$ with $p \in [1,+\infty]$ and $(a,b) \subset \mathbb{R}$. Let $c \in \mathbb{R}$. Prove that the function $v(x) := u(x-c)$ is in $W^{1,p}((a+c,b+c))$.*

Exercice 2. *Let $u \in W^{1,p}((a,b))$ with $p \in [1,+\infty]$. Suppose that $\eta \in C^1(a,b) \cap C([a,b])$ and suppose that η' is bounded on (a,b).*

(1) Prove that $u\eta \in W^{1,p}((a,b))$;

(2) Prove that if $\eta(a) = \eta(b) = 0$, then

$$\int_a^b u(x)\eta'(x)\,dx = -\int_a^b u'(x)\eta(x)\,dx.$$

(3) Prove that if $\eta(a) = \eta(b) = 0$, then the function $u\eta$, extended as zero outside (a,b), is in $W^{1,p}(\mathbb{R})$.

Exercice 3. *Let $u \in W^{1,p}((a,b))$ with $p \in [1,+\infty]$. Suppose that $u(a) = u(b) = 0$. Prove that the function*

$$\widetilde{u}(x) = \begin{cases} u(x) & \text{if} \quad x \in (a,b), \\ 0 & \text{if} \quad x \notin (a,b), \end{cases}$$

is in $W^{1,p}(\mathbb{R})$.

Exercice 4. *Consider the interval $(a,b) \subset \mathbb{R}$ and a function $u \in W^{1,p}((a,b))$. Using the second point of Exercice 2, prove that the function*

$$\widetilde{u}(x) = \begin{cases} u(x) & \text{if} \quad x \in (a,b), \\ u(2b-x) & \text{if} \quad x \notin (b,2b-a), \end{cases}$$

is in $W^{1,p}((a,2b-a))$.

Exercice 5. *Let $u \in W^{1,p}((a,b))$ with $p \in [1,+\infty]$. Suppose that for every $\varphi \in C_c^\infty(\mathbb{R})$ (so the support of φ is not necessarily contained in (a,b)) we have*

$$\int_a^b u(x)\varphi'(x)\,dx = -\int_a^b u'(x)\varphi(x)\,dx.$$

Prove that the function

$$\widetilde{u}(x) = \begin{cases} u(x) & \text{if} \quad x \in (a,b), \\ 0 & \text{if} \quad x \notin (a,b), \end{cases}$$

is in $W^{1,p}(\mathbb{R})$.

Exercice 6. *Let $u \in W^{1,p}((-\infty,a))$ and $u \in W^{1,p}((a,+\infty))$ with $a \in \mathbb{R}$ and $p \in [1,+\infty]$. Prove that the following are equivalent:*

(1) $u(a) = v(a)$;

(2) the function

$$w(x) = \begin{cases} u(x) & \text{if} \quad x \in (-\infty,a), \\ v(x) & \text{if} \quad x \in (a,+\infty), \end{cases}$$

is in $W^{1,p}(\mathbb{R})$.