Exercices

Exercice 1 (Product of Banach spaces)**.** *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two Banach spaces. Prove that the product space $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$ equipped with the norm*

$$\|(u,v)\|_{\mathcal{B}} = \|u\|_{\mathcal{B}_1} + \|v\|_{\mathcal{B}_2}$$

is a Banach space. Moreover, if $\mathcal{B}_1$ and $\mathcal{B}_2$ are separable, then also $\mathcal{B}$ is separable.

Exercice 2 (Weak topology on a product space)**.** *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two Banach spaces and let $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$ be the Banach space equipped with the norm*

$$\|(u,v)\|_{\mathcal{B}} = \|u\|_{\mathcal{B}_1} + \|v\|_{\mathcal{B}_2}.$$

Prove that any functional $T \in \mathcal{B}'$ can be written in the form

$$T(u,v) = T_1(u) + T_2(v),$$

where $T_1 \in \mathcal{B}_1'$ and $T_2 \in \mathcal{B}_2'$.

Exercice 3 (Product of reflexive spaces)**.** *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two reflexive Banach spaces. Is it true that $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$ is reflexive?*

Exercice 4. *Given $p, r \in [1, +\infty]$ and an open interval $I \subset \mathbb{R}$, consider the space $\mathcal{B}$ of all functions $u \in L^p(I)$ for which there exists $v \in L^r(I)$ such that*

$$\int_I u\varphi' = -\int_I v\varphi \quad \textit{for all} \quad \varphi \in C^1_c(I).$$

1. *Prove that the function v with the above property is unique. We will denote this function by u'.*
2. *Prove that $\mathcal{B}$ is a Banach space with the norm*
$$\|u\|_{L^p(I)} + \|u'\|_{L^r}.$$
3. *Prove that, when $p, r \in [1, +\infty)$, every functional in the dual space $\mathcal{B}'$ is of the form*
$$T(u) = \int_I u\varphi + \int_u' \psi$$
for some $\varphi \in L^{p'}(I)$ and $\psi \in L^{r'}(I)$.
4. *Prove that, when $p, r \in (1, +\infty)$, the following are equivalent for a sequence $u_n \in \mathcal{B}$:*
   - *there is $u \in \mathcal{B}$ such that $u_n \rightharpoonup u$ in $\mathcal{B}$;*
   - *there is $u \in \mathcal{B}$ such that $u_n \rightharpoonup u$ in L^p and $u_n' \rightharpoonup u'$ in L^r;*
   - *there are $u \in L^p$ and $v \in L^r$ such that $u_n \rightharpoonup u$ in L^p and $u_n' \rightharpoonup v$ in L^r.*
5. *Prove that, when $p, r \in (1, +\infty)$, any bounded sequence $u_n \in \mathcal{B}$ admits a weakly convergent subsequence.*
6. *Explore the above question for all possible combination of $p, r \in [1, +\infty]$:*
   - *Is it true that, when $p \in (1, +\infty)$ and $r = 1$, any bounded sequence $u_n \in \mathcal{B}$ admits a weakly convergent subsequence?*

- *Is it true that, when $p \in (1, +\infty)$ and $r = +\infty$, any bounded sequence $u_n \in \mathcal{B}$ admits a weakly convergent subsequence?*
- *Is it true that, when $p = 1$ and $r \in (1, +\infty)$, any bounded sequence $u_n \in \mathcal{B}$ admits a weakly convergent subsequence?*
- *Is it true that, when $p = +\infty$ and $r \in (1, +\infty)$, any bounded sequence $u_n \in \mathcal{B}$ admits a weakly convergent subsequence?*
- *Is it true that, when $p = 1$ and $r = 1$, any bounded sequence $u_n \in \mathcal{B}$ admits a weakly convergent subsequence?*
- *Is it true that, when $p = 1$ and $r = +\infty$, any bounded sequence $u_n \in \mathcal{B}$ admits a weakly convergent subsequence?*
- *Is it true that, when $p = +\infty$ and $r = 1$, any bounded sequence $u_n \in \mathcal{B}$ admits a weakly convergent subsequence?*
- *Is it true that, when $p = +\infty$ and $r = +\infty$, any bounded sequence $u_n \in \mathcal{B}$ admits a weakly convergent subsequence?*