Exercices and examples

Exercice 1. *Let I be an open interval in $\mathbb{R}$ and let $p \in (1, +\infty)$.*
Let $u_n \in L^p(I)$ be a sequence converging weakly in $L^p(I)$ to a function $u \in L^p(I)$.

(1) *Is it true that $|u_n|$ converges weakly in $L^p(I)$ to $|u|$?*
(2) *Is it true that u_n^+ converges weakly in $L^p(I)$ to u^+?*
(3) *Is it true that, up to a subsequence, $|u_n|$ converges weakly in $L^p(I)$ to some function $v \in L^p(I)$?*
(4) *Is it true that the whole sequence $|u_n|$ converges weakly in $L^p(I)$ to some function $v \in L^p(I)$?*
(5) *Fix a function $\varphi \in C_c^\infty(I)$ and a sequence of bounded measurable functions*
$$v_n : I \to [-1, 1],$$
converging pointwise almost-everywhere to some
$$v : I \to [-1, 1].$$
Is it true that
$$\int_I u_n \varphi v_n \to \int_I u \varphi v\,?$$
(6) *Suppose that*
$$-1 \le u_n \le 1 \quad \text{for all} \quad n \in \mathbb{N}.$$
Is it true that u_n^2 converges weakly to u^2?

Exercice 2. *Let I be an open interval in $\mathbb{R}$ and let $p \in (1, +\infty)$.*
Let $u_n \in L^p(I)$ be a sequence converging weakly in $L^p(I)$ to a function $u \in L^p(I)$. Suppose that $u_n \in W^{1,p}(I)$ for every n and that the sequence of norms $\|u_n\|_{W^{1,p}(I)}$ is bounded.

(1) *Is it true that $u \in W^{1,p}(I)$?*
(2) *Is it true that u_n converges weakly in $W^{1,p}(I)$ to u?*
(3) *Is it true that $|u_n|$ converges weakly in $W^{1,p}(I)$ to $|u|$?*
(4) *Is it true that u_n^+ converges weakly in $W^{1,p}(I)$ to u^+?*
(5) *Is it true that u_n^2 converges weakly in $W^{1,p}(I)$ to u^2?*

Exercice 3. *Let I be an open interval in $\mathbb{R}$ and let $p \in (1, +\infty)$.*
Let $u_n \in W^{1,p}(I)$ be a sequence converging weakly in $W^{1,p}(I)$ to some $u \in W^{1,p}(I)$.

(1) *Show that if I is bounded, then u_n converges to u uniformly on I.*
(2) *Show that if I is unbounded, then $u_n(x)$ converges to $u(x)$ for every $x \in I$.*