Exercices

Esercizio 1. *Let $\mathcal{H}$ be an Hilbert space. If $x_n \to x$ in $\mathcal{H}$ and $y_n \rightharpoonup y$ in $\mathcal{H}$, then*

$$< x, y >= \lim_{n\to\infty} < x_n, y_n > .$$

Esercizio 2. *Let $\Omega = [0, 1]$. Find two sequences in $L^2(\Omega)$ such that:*

- $f_n \rightharpoonup f \in L^2(\Omega)$;
- $g_n \rightharpoonup g \in L^2(\Omega)$;
- $\displaystyle\int_\Omega f(x)g(x)\,dx \neq \lim_{n\to+\infty} \int_\Omega f_n(x)g_n(x)\,dx.$

Esercizio 3. *Let $\Omega \subset \mathbb{R}^d$ be a measurable set and $p \in (1, +\infty)$. Show that if $u_n \rightharpoonup u$ in $L^p(\Omega)$ and $v \in L^\infty(\Omega)$, then $u_n v \rightharpoonup uv$ in $L^p(\Omega)$.*

Esercizio 4. *Let $p \in (1, +\infty)$ and $f \in L^p(\mathbb{R}^d)$. Consider a sequence $R_n \to +\infty$. For every $n \geq 1$, let*

$$f_n(x) := \frac{1}{R_n^{d/p}} f\left({}^{x}/_{R_n}\right).$$

Show that $f_n \rightharpoonup 0$ in $L^p(\mathbb{R}^d)$.

Esercizio 5. *Let $p \in (1, +\infty)$. Show that if $u_n \rightharpoonup u$ in $L^p(\mathbb{R}^d)$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$, then $u_n * \varphi \rightharpoonup u * \varphi$ in $L^p(\mathbb{R}^d)$.*

Esercizio 6. *Let $\Omega \subset \mathbb{R}^d$ be a measurable set and let $p, q, r \in (1, +\infty)$ be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Show that if $u_n \to u$ strongly in $L^p(\Omega)$ and $v_n \rightharpoonup v$ weakly in $L^q(\Omega)$, then $u_n v_n \rightharpoonup uv$ weakly in $L^r(\mathbb{R}^d)$.*

Esercizio 7. *Let $\Omega \subset \mathbb{R}^d$ be a measurable set and let $p, q, r \in (1, +\infty)$ be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Is it true that the weak convergences $u_n \rightharpoonup u$ in $L^p(\Omega)$ and $v_n \rightharpoonup v$ in $L^q(\Omega)$, imply that $u_n v_n \rightharpoonup uv$ weakly in $L^r(\mathbb{R}^d)$?*

Esercizio 8. *Let $\Omega \subset \mathbb{R}^d$ be a bounded measurable set. Let $u_n \in L^2(\Omega)$ be a sequence of functions such that, for all $n \geq 1$, $0 \leq u_n \leq 1$ on Ω. Show that if u_n converges weakly to 0 in $L^2(\Omega)$, then u_n converges strongly to zero in $L^2(\Omega)$.*

Esercizio 9. *Let $\Omega \subset \mathbb{R}^d$ be a measurable set and let $p \in (1, +\infty)$. Is it true that for every sequence $u_n \rightharpoonup u$ in $L^p(\Omega)$, there exists a subsequence u_n that converges pointwise almost-everywhere to u?*

Esercizio 10. *Let $p \in (2, +\infty)$ and $u_n \in L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ be a sequence that converges weakly in $L^p(\mathbb{R}^d)$ to a function $u \in L^p(\mathbb{R}^d)$. Prove that the following are equivalent:*

(1) $u \in L^2(\mathbb{R}^d)$;

(2) u_n *converges weakly in* $L^2(\mathbb{R}^d)$ *to some function* $v \in L^2(R^d)$;

(3) $u \in L^2(\mathbb{R}^d)$ *and* u_n *converges weakly in* $L^2(\mathbb{R}^d)$ *to* u;

(4) u_n *is bounded in* $L^2(\mathbb{R}^d)$.

Esercizio 11. *Let $\Omega \subset \mathbb{R}^d$ be a measurable set and let $p \in [1, +\infty]$. Show that if $u_n \in L^p(\Omega)$ converges weakly to $u \in L^p(\Omega)$ and along the sequence we have $0 \leq u_n \leq 1$ for all $n \geq 1$, then also $0 \leq u \leq 1$.*

Esercizio 12. *Let $u_n \in L^2(0, 1)$ be a sequence that converges weakly to $u \in L^2(0, 1)$. Is it true that $\sin(u_n)$ converges weakly to $\sin(u)$ in $L^2(0, 1)$?*

Esercizio 13 (Written exam 2002, P. Acquistapace). *Let* $f_n \in L^2(0,1)$ *and* $g_n \in L^\infty(0,1)$ *be two sequences such that:*

- $f_n \rightharpoonup 0$ *in* $L^2(0,1)$*;*
- $g_n \to 0$ *pointwise almost-everywhere on* $[0,1]$*;*
- g_n *is bounded in* $L^\infty(0,1)$*.*

Prove that $f_n g_n \rightharpoonup 0$ *in* $L^2(0,1)$*.*

Esercizio 14 (Written exam 2002, P. Acquistapace). *Find two sequences* $f_n \in L^2(0,1)$ *and* $g_n \in L^\infty(0,1)$ *such that:*

- $f_n \rightharpoonup 0$ *in* $L^2(0,1)$*;*
- $g_n \to 0$ *pointwise almost-everywhere on* $[0,1]$*;*

and such that $f_n g_n$ *does not converge weakly to* 0 *in* $L^2(0,1)$*.*

Esercizio 15 (Written exam 2003, P. Acquistapace). *Find two sequences* $f_n \in L^2(0,1)$ *and* $g_n \in L^\infty(0,1)$ *such that:*

- $f_n \rightharpoonup 0$ *in* $L^2(0,1)$*;*
- $g_n \to 0$ *pointwise almost-everywhere on* $[0,1]$*;*

and such that $f_n g_n$ *does not converge weakly to* 0 *in* $L^2(0,1)$*.*