An example of a Banach space which is not a dual space

Proposition 1. *The space $L^1((0,1))$ is not the dual space of any Banach space.*

Proof. Suppose that $\mathcal{B}$ is a Banach space such that its dual $\mathcal{B}'$ is precisely $L^1((0,1))$. Let $\overline{B}_1'$ be the unit ball in $\mathcal{B}'$:

$$\overline{B}_1' = \Big\{T \in \mathcal{B}' \ :\ \|T\|_{\mathcal{B}'} \le 1\Big\}.$$

Given a set

$$E \subset \overline{B}_1'$$

we will say that E is *extremal* if it has the following property:

$$(1)\qquad \begin{cases} \textit{if } S,T \in \overline{B}_1' \textit{ and if there is } t \in (0,1) \textit{ such that } (1-t)S + tT \in E, \\ \qquad\qquad\qquad \text{then } S \in E \text{ and } T \in E. \end{cases}$$

Consider the family Σ of all sets $E \subset \overline{B}_1'$ such that:

- E is extremal;
- E is closed with respect to the weak-$*$ topology $\sigma(\mathcal{B}', \mathcal{B})$;
- E is non-empty.

Consider the following order relation on Σ:

$$E_1 < E_2 \quad \text{if} \quad E_2 \subset E_1.$$

Step 1. *Existence of maximal elements.* Let Γ be a totally ordered subset of Σ. Then, the intersection

$$\bigcap_{E\in\Gamma} E.$$

is in Σ. Indeed:

- since the sets $E \in \Gamma$ are closed also their intersection is closed in the weak-$*$ topology;
- suppose that $S,T \in \overline{B}_1'$ and that there is $t \in (0,1)$ such that

  $$(1-t)S + tT \in \bigcap_{E\in\Gamma} E.$$

  Since every $E \in \Gamma$ has the property (1), we have that $S,T \in E$ for all $E \in \Gamma$. Then,

  $$S,T \in \bigcap_{E\in\Gamma} E.$$

- Finally, we notice that since all the sets $E \in \Gamma$ are non-empty and closed in the weak-$*$ topology, and since $\overline{B}_1'$ is compact in the weak-$*$ topology, the intersection $\bigcap_{E\in\Gamma} E$ is non-empty.

By the Zorn's lemma, there is a set $\widetilde{E} \in \Sigma$ which is maximal with respect to the order relation $<$.

Step 2. *The set $\widetilde{E}$ has only one element.* Suppose by contradiction that there are

$$S,T \in \widetilde{E} \quad \text{such that} \quad S \neq T.$$

Then, there is an element

$$x \in \mathcal{B} \quad \text{such that} \quad S(x) \neq T(x).$$

Consider the set

$$\widetilde{E}' = \Big\{T \in \widetilde{E} \ :\ T(x) = \sup_{\tau\in\widetilde{E}} \tau(x)\Big\}.$$

$\widetilde{E}'$ has the following properties:

- $\widetilde{E}'$ *is non-empty.*
  Let $T_n \in \widetilde{E}$ be a maximizing sequence:
  $$\lim_{n\to+\infty} T_n(x) = \sup_{\tau\in\widetilde{E}} \tau(x).$$
  Since $\mathcal{B}' = L^1(0,1)$ is separable, we have that $\mathcal{B}$ is separable, so that the unit ball $\overline{B}_1'$ is compact with respect to the weak-$*$ convergence. Then, there is a subsequence T_{n_k} and $T_\infty \in \widetilde{E}$ such that
  $$T_{n_k} \rightharpoonup^* T_\infty.$$
  Then,
  $$T_\infty(x) = \lim_{k\to+\infty} T_{n_k}(x) = \sup_{\tau\in\widetilde{E}} \tau(x).$$

- $\widetilde{E}'$ *is closed with respect to the weak-$*$ topology.*
  Indeed, set
  $$\lambda := \sup_{\tau\in\widetilde{E}} \tau(x).$$
  Then, the set $\{S \in \mathcal{B}' \ :\ S(x) = \lambda\}$ is closed with respect to the weak-$*$ topology. Then,
  $$\widetilde{E}' = \widetilde{E} \cap \{S \in \mathcal{B}' \ :\ S(x) = \lambda\}$$
  is also closed in the weak-$*$ topology.

- $\widetilde{E}'$ *is extremal in the sense of* (1)*.*
  Suppose that $T_1, T_2 \in \overline{B}_1'$ and that there exists $t \in (0,1)$ such that
  $$(1-t)T_1 + tT_2 \in \widetilde{E}'.$$
  Since $\widetilde{E}' \subset \widetilde{E}$, the extramality property of $\widetilde{E}$ implies that $T_1, T_2 \in \widetilde{E}$. Then, since
  $$\sup_{\tau\in\widetilde{E}} \tau(x) = (1-t)T_1(x) + tT_2(x) \le (1-t)\sup_{\tau\in\widetilde{E}} \tau(x) + t\sup_{\tau\in\widetilde{E}} \tau(x) = \sup_{\tau\in\widetilde{E}} \tau(x),$$
  so we have the identities
  $$T_1(x) = T_2(x) = \sup_{\tau\in\widetilde{E}} \tau(x).$$
  Thus, by definition $T_1, T_2 \in \widetilde{E}'$, which proves that $\widetilde{E}'$ is in Σ.

- $\widetilde{E}'$ *is a proper subset of* $\widetilde{E}$*.*
  Indeed, since $S(x) \neq T(x)$ it is not possible that both S and T are in $\widetilde{E}'$.

Since by assumption $\widetilde{E}$ is maximal, this is a contradiction. In conclusion $\widetilde{E}$ contains only one element.

Step 3. *If a set $\widetilde{E} \subset L^1(0,1)$ has only one element, then it cannot be extremal.* Suppose that $f \in L^1(0,1)$ is the only element of $\widetilde{E}$. We consider two cases:

- If $\|f\|_{L^1} < 1$, then we can find a function $\phi \in L^1$, $\phi \neq 0$, such that
  $$\|f+\phi\|_{L^1} < 1 \quad \text{and} \quad \|f-\phi\|_{L^1} < 1.$$
  Since ,
  $$f = \frac{1}{2}(f+\phi) + \frac{1}{2}(f-\phi),$$
  we get that $\widetilde{E}$ is not extremal.

- If $\|f\|_{L^1} = 1$, then we can find $a \in (0,1)$ such that

$$\int_0^a |f| = \int_a^1 |f| = \frac{1}{2},$$

and consider the functions

$$f_1 := 2f \inf_{[0,a]} \quad \text{and} \quad f_2 := 2f \inf_{[a,1]} .$$

Then,

$$f = \frac{1}{2} f_1 + \frac{1}{2} f_2 \ , \quad \|f_1\|_{L^1} = 1 \ , \quad \|f_2\|_{L^1} = 1.$$

This prove that $\widetilde{E}$ is not extremal, which concludes the proof.

$\square$