On the separability of $\mathcal{B}$ and $\mathcal{B}'$

Proposition 1. *Let $\mathcal{B}$ be a Banach space and let $\mathcal{B}'$ be its dual.*
If $\mathcal{B}'$ is separable, then also $\mathcal{B}$ is separable.

Proof. Let $(T_n)_{n\geq 1}$ be a dense (with respect to the topology induced by the norm $\|\cdot\|_{\mathcal{B}'}$) subset of $\mathcal{B}'$. By the definition of $\|T_n\|_{\mathcal{B}'}$, for every T_n, we can find $x_n \in \mathcal{B}$ such that:

$$\|x_n\|_{\mathcal{B}} = 1 \qquad \text{and} \qquad \frac{1}{2}\|T_n\|_{\mathcal{B}'} \leq T_n(x_n) \leq \|T_n\|_{\mathcal{B}'}.$$

Consider the countable set

$$V_{\mathbb{Q}} = \Big\{ \sum_{i=1}^{N} x_i q_i \ : \ q_i \in \mathbb{Q},\ N \geq 1 \Big\},$$

and its closure V in $\mathcal{B}$ with respect to the strong topology of $\mathcal{B}$. Then V is a closed linear subspace of $\mathcal{B}$. Suppose that $V \neq \mathcal{B}$. Then, by the Hahn-Banach's theorem, there is a functional $T \in \mathcal{B}'$ such that

$$\|T\|_{\mathcal{B}'} = 1 \quad \text{and} \quad T \equiv 0 \ \text{ on } \ V.$$

Since $(T_n)_{n\geq 1}$ is dense we can find a sequence $(T_{n_k})_{k\geq 1}$ such that

$$\lim_{k\to+\infty} \|T_{n_k} - T\|_{\mathcal{B}'} = 0.$$

We then have

$$\begin{aligned}\frac{1}{2}\|T_{n_k}\|_{\mathcal{B}'} \leq |T_{n_k}(x_{n_k})| &= |T_{n_k}(x_{n_k}) - T(x_{n_k})| \\ &\leq \|T_{n_k} - T\|_{\mathcal{B}'}\|x_{n_k}\|_{\mathcal{B}} \\ &\leq \|T_{n_k} - T\|_{\mathcal{B}'}.\end{aligned}$$

On the other hand

$$\lim_{k\to+\infty} \|T_{n_k}\|_{\mathcal{B}'} = 1 \qquad \text{and} \qquad \lim_{k\to+\infty} \|T_{n_k} - T\|_{\mathcal{B}'} = 0,$$

which is a contradiction. Thus $\mathcal{B}$ is the closure of $V_{\mathbb{Q}}$. □

Exercise 2. *Find an example of a Banach space $\mathcal{B}$, which is separable, but its dual $\mathcal{B}'$ is not.*