The weak-$*$ topology on the dual of a Banach space

The weak-$*$ topology on $\mathcal{B}'$

Let $\mathcal{B}$ be a Banach space and let $\mathcal{B}'$ be its dual.

Definition 1 (Weak-$*$ topology)**.** *The weak topology on $\mathcal{B}'$ is the topology $\sigma(\mathcal{B}',\mathcal{B})$, which is:*

the weakest topology on $\mathcal{B}'$ that makes all the evaluation maps in $\mathcal{B}\subset\mathcal{B}''$ continuous.

Remark 2 (Strong topology)**.** *The strong topology on $\mathcal{B}'$ is the topology induced by the norm $\|\cdot\|_{\mathcal{B}'}$.*

Remark 3 (Weak-$*$ topology and weak-$*$ convergence)**.** *Let T_n be a sequence in $\mathcal{B}'$ and let $T\in\mathcal{B}'$. Then, the following are equivalent:*

(1) *$T_n\to T$ in the topology of $\sigma(\mathcal{B}',\mathcal{B})$;*

(2) *$T_n(x)\to T(x)$, for every $x\in\mathcal{B}$.*

We will say that T_n converges to T weakly-$*$ in $\mathcal{B}'$ and we will write $T_n\rightharpoonup^* T$.

Proposition 4 (The weak-$*$ topology is Hausdorff)**.** *Let $\mathcal{B}$ be a Banach space with dual $\mathcal{B}'$ and let $\sigma(\mathcal{B}',\mathcal{B})$ be the weak-$*$ topology on $\mathcal{B}'$. Then, for every couple of distinct operators $S,T\in\mathcal{B}'$, there are open sets $V,W\in\sigma(\mathcal{B}',\mathcal{B})$ such that:*

$$S\in V\ ,\quad T\in W\ ,\quad V\cap W=\emptyset.$$

Proof. Since $S\neq T$ as operators on $\mathcal{B}$, there is an element $x\in\mathcal{B}$ such that $S(x)\neq T(x)$. □

Proposition 5 (Boundedness and semicontinuity of the operator norm)**.** *Let T_n be a sequence in $\mathcal{B}'$ and let $T\in\mathcal{B}'$. If $T_n\rightharpoonup^* T$, then the sequence of norms $\|T_n\|_{\mathcal{B}'}$ is bounded and*

$$\|T\|_{\mathcal{B}'}\le\liminf_{n\to+\infty}\|T_n\|_{\mathcal{B}'}.$$

Proof. The sequence of norms is bounded thanks to Banach-Steinhaus. In order to prove the lower-semicontinuity of the norms, we consider a subsequence T_{n_k} such that

$$\liminf_{k\to+\infty}\|T_{n_k}\|_{\mathcal{B}'}=\liminf_{n\to+\infty}\|T_n\|_{\mathcal{B}'}.$$

We next fix $x\in\mathcal{B}$, with $\|x\|_{\mathcal{B}}\le 1$, and we compute

$$T(x)=\lim_{k\to+\infty}T_{n_k}(x)\le\lim_{k\to+\infty}\|T_{n_k}\|_{\mathcal{B}'}\|x\|_{\mathcal{B}}=\liminf_{n\to+\infty}\|T_n\|_{\mathcal{B}'}.$$

Taking the supremum over all $x\in\mathcal{B}$, with $\|x\|_{\mathcal{B}}\le 1$, we get

$$\|T\|_{\mathcal{B}'}\le\liminf_{n\to+\infty}\|T_n\|_{\mathcal{B}'}.$$

□

Proposition 6. *Let x_n be a sequence in $\mathcal{B}$ and let $x\in\mathcal{B}$. Let T_n be a sequence in $\mathcal{B}'$ and let $T\in\mathcal{B}'$. If*

$$x_n\to x\qquad\textit{and}\qquad T_n\rightharpoonup^* T,$$

then

$$T(x)=\lim_{n\to+\infty}T_n(x_n).$$

Proof. We write

$$\begin{aligned}|T_n(x_n) - T(x)| &= |T_n(x_n) - T_n(x)| + |T_n(x) - T(x)| \\ &\le \|T_n\|_{\mathcal{B}'}\|x_n - x\|_{\mathcal{B}} + |T_n(x) - T(x)|.\end{aligned}$$

By the strong convergence of x_n to x, we have $\|x_n - x\|_{\mathcal{B}} \to 0$, while the weak convergence of x_n gives that $\|T_n\|_{\mathcal{B}'}$ is bounded. Thus,

$$\|x_n - x\|_{\mathcal{B}}\|T_n\|_{\mathcal{B}'} \to 0.$$

On the other hand, the weak-$*$ convergence of T_n gives

$$|T_n(x) - T(x)| \to 0,$$

which concludes the proof. □

Weak-$*$ compactness of the unit ball in $\mathcal{B}'$

Theorem 7 (Tychonoff). *Let $\{K_i\}_{i\in\mathcal{I}}$, be a family of compact spaces, where $\mathcal{I}$ is any set of indices. Consider the product space $K = \prod_{i\in\mathcal{I}} K_i$ and the family of projection maps*

$$\pi_j : K \to K_j \ , \quad \pi_i\Big((\omega_i)_{i\in\mathcal{I}}\Big) = \omega_j.$$

On the product space K we consider the topology σ, which is the weakest topology that makes all the maps $(\pi_j)_{j\in\mathcal{I}}$ continuous. Then, (K,σ) is compact.

Proof. Let $\mathcal{F}$ be an infinite family of closed sets (in the product topology of K) with the following property:

$$(\bigstar) \text{ for any finite family } C_1,\dots,C_n \in \mathcal{F} \text{ we have } C_1 \cap \dots \cap C_n \neq \emptyset.$$

In order to prove that K is compact, it is sufficient to show that

$$\bigcap_{C\in\mathcal{F}} C \neq \emptyset.$$

Step 1. Consider the collection Γ of all families of sets (closed or not) containing all the sets from $\mathcal{F}$ and satisfying $(\bigstar)$. For two families $\mathcal{F}_1$ and $\mathcal{F}_2$ we say that $\mathcal{F}_1 < \mathcal{F}_2$ if all the elements of $\mathcal{F}_1$ are elements of $\mathcal{F}_2$. We notice that Γ admits maximal families. Indeed, by Zorn's lemma, it is sufficient to prove that every totally ordered subset Σ of Γ admits a maximal element. Indeed, it is sufficient to take

$$\mathcal{F}_* = \bigcup_{\mathcal{F}\in\Sigma} \mathcal{F}.$$

We notice that $\mathcal{F}_*$ still has the property $(\bigstar)$ since any finite family $C_1,\dots,C_N \in \mathcal{F}_*$ belongs to some element $\mathcal{F}$ of Σ.

Step 2. Let $\mathcal{F}_*$ be a maximal family satisfying $(\bigstar)$ and containing $\mathcal{F}$. Then, $\mathcal{F}_*$ has the following properties:

- if $C_1,\dots,C_n \in \mathcal{F}_*$, then $\displaystyle\bigcap_{i=1}^n C_i \in \mathcal{F}_*$;
- if $\widetilde{C}$ is such that $\widetilde{C} \cap C \neq \emptyset$ for all $C \in \mathcal{F}_*$, then $\widetilde{C} \in \mathcal{F}_*$;
- for each $i \in \mathcal{I}$ the following family of subsets of K_i

  $$\Big\{\pi_i(C) \ : \ C \in \mathcal{F}_*\Big\}$$

  has the property $(\bigstar)$. Since K_i is compact, there is a point $\omega_i \in K_i$ such that

  $$\omega_i \in \overline{\pi_i(C)} \quad \text{for all} \quad C \in \mathcal{F}_*.$$

Step 3. We will show that $\omega := (\omega_i)_{i\in\mathcal{I}}$ lies in all $\overline{C}$ with $C \in \mathcal{F}_*$. Let $A \subset K$ be any neighborhood of ω of the form

$$A = \bigcap_{j=1}^{N} \pi_j^{-1}(A_j),$$

where $A_j \subset K_j$ are open sets. Now, since

$$\omega_i \in \overline{\pi_i(C)} \quad \text{for all} \quad C \in \mathcal{F}_*,$$

and since $\omega_i \in A_i$ we have that

$$A_i \cap \overline{\pi_i(C)} \neq \emptyset \quad \text{for all} \quad C \in \mathcal{F}_*.$$

since A_i is open we have

$$A_i \cap \pi_i(C) \neq \emptyset \quad \text{for all} \quad C \in \mathcal{F}_*.$$

Then,

$$\pi_i^{-1}(A_i) \cap C \neq \emptyset \quad \text{for all} \quad C \in \mathcal{F}_*.$$

This implies that

$$\pi_i^{-1}(A_i) \in \mathcal{F}_*,$$

and so we also get

$$A \in \mathcal{F}_*.$$

In particular, this implies that

$$A \cap C \neq \emptyset \quad \text{for every} \quad C \in \mathcal{F}_*,$$

and so

$$A \cap C \neq \emptyset \quad \text{for every} \quad C \in \mathcal{F}.$$

Since A is arbitrary, we get that

$$\omega \in C \neq \emptyset \quad \text{for every} \quad C \in \mathcal{F}.$$

$\square$

Theorem 8 (Banach-Alaoglu-Bourbaki). *Let $\mathcal{B}$ be a Banach space and let $\mathcal{B}'$ be its dual. Then, the unit ball*

$$B_1' = \Big\{T \in \mathcal{B}' \;:\; \|T\|_{\mathcal{B}'} \leq 1\Big\},$$

is compact in the weak-$*$ topology $\sigma(\mathcal{B}', \mathcal{B})$.

Proof. Consider as family of indices $\mathcal{I}$ all the points of $\mathcal{B}$. For every $x \in \mathcal{B}$, let K_x be the compact interval

$$K_x = \big[-\|x\|_{\mathcal{B}}, \|x\|_{\mathcal{B}}\big].$$

Consider the product space

$$K = \prod_{x\in\mathcal{B}} K_x,$$

equipped with the product topology σ that makes all the projection maps

$$\pi_x : K \to K_x\ , \qquad \pi_x(\omega) = \omega_x,$$

continuous. Notice that every map

$$T : \mathcal{B} \to \mathbb{R}\ , \quad \|T\|_{\mathcal{B}'} \leq 1$$

corresponds to a point

$$\omega \in K\ , \quad \omega = \Big(T(x)\Big)_{x\in\mathcal{B}}.$$

Indeed, it is sufficient to notice that, for any fixed $x, y \in \mathcal{B}$ and $t \in \mathbb{R}$, the sets

$$\Big\{\omega \in K \;:\; \pi_{x+y}(\omega) - \pi_x(\omega) - \pi_y(\omega) = 0\Big\}$$

$$\Big\{\omega \in K \;:\; \pi_{tx}(\omega) - t\pi_x(\omega) = 0\Big\}$$

are closed in K and that the intersection of all such sets (over all $x, y \in \mathcal{B}$ and all $t \in \mathbb{R}$) give precisely the set of points $\omega \in K$ corresponding to linear maps $T : \mathcal{B} \to \mathbb{R}$ with $\|T\|_{\mathcal{B}'} \leq 1$. □

Corollary 9. *Let $\mathcal{B}$ be a Banach space and let $\mathcal{B}'$ be its dual. Suppose that $\mathcal{B}$ is reflexive. Then, the unit ball*

$$B_1 = \Big\{x \in \mathcal{B} \;:\; \|x\|_{\mathcal{B}} \leq 1\Big\},$$

is compact in the weak topology $\sigma(\mathcal{B}, \mathcal{B}')$.

Proof. Follows from the Banach-Alaoglu-Bourbaki's theorem applied to the Banach space $\mathcal{B}'$ and its dual $\mathcal{B}''$ (which by hypothesis coincides with $\mathcal{B}$). □

Finally, we notice that also the converse is true. In fact, we have the following theorem.

Theorem 10 (Kakutani)**.** *Let $\mathcal{B}$ be a Banach space and let $\mathcal{B}'$ be its dual. Suppose that the unit ball*

$$B_1 = \Big\{x \in \mathcal{B} \;:\; \|x\|_{\mathcal{B}} \leq 1\Big\},$$

is compact in the weak topology $\sigma(\mathcal{B}, \mathcal{B}')$. Then, $\mathcal{B}$ is reflexive.

Proof. See theorem □

Metrizability of the weak-$*$ topology on the unit ball of $\mathcal{B}'$

Proposition 11. *Let $\mathcal{B}$ be a separable Banach space and let $\mathcal{B}'$ be its dual.*
Then, the weak-$*$ topology $\sigma(\mathcal{B}', \mathcal{B})$ on the unit ball

$$\overline{B}'_1 = \Big\{T \in \mathcal{B}' \;:\; \|T\|_{\mathcal{B}'} \leq 1\Big\},$$

is metrizable.

Consider a countable dense subset $(x_n)_{n\geq 1}$ of $\mathcal{B}$ and set

$$\widetilde{x}_n := \frac{1}{\|x_n\|_{\mathcal{B}}} x_n \in \mathcal{B}.$$

Given $S, T \in \mathcal{B}'$, define

$$\delta(S, T) := \sum_{n=1}^{+\infty} \frac{1}{2^n} |S(\widetilde{x}_n) - T(\widetilde{x}_n)|.$$

Exercise 12. *Prove that δ is a metric on $\mathcal{B}'$.*

Exercise 13. *In B'_1, the topology induced by δ is equivalent to $\sigma(\mathcal{B}', \mathcal{B})$.*