The weak topology on a Banach space

A general construction

Let X be a set and let $\mathcal{F}$ be a collection of functions

$$f : \mathcal{F} \to \mathbb{R}.$$

We consider all the subsets

$$A \subset X$$

of the form

$$A = \bigcap_{i=1}^{N} f_i^{-1}(\omega_i),$$

where:

- $N \in \mathbb{N}$;
- $f_i \in \mathcal{F}$, $i = 1, \ldots, N$, is a collection of maps;
- $\omega_i \subset \mathbb{R}$, $i = 1, \ldots, N$, is a collection of open sets.

For simplicity, if $A \subset W$ is as above, we will say that

$$A \in \mathcal{N}(X, \mathcal{F}).$$

Proposition 1. *Let now $\sigma(X, \mathcal{F})$ be the collection of all sets A of the form*

$$A = \bigcup_{i \in I} A_i,$$

where:

- *$\mathcal{I}$ is a finite or infinite set of indices;*
- *for each $i \in \mathcal{I}$, $A_i \in \mathcal{N}(X, \mathcal{F})$.*

Then, $\sigma(X, \mathcal{F})$ is a topology on X.

Remark 2. Every map $f \in \mathcal{F}$ is continuous with respect to $\sigma(X, \mathcal{F})$. Moreover, if τ is a topology on X for which all the maps $f \in \mathcal{F}$ are continuous, then τ contains $\sigma(X, \mathcal{F})$. That is,

$\sigma(X, \mathcal{F})$ is the *coarsest topology*
(said also *weakest topology*, *most economical topology*, *least fine topology*)
on X that makes all the maps $f \in \mathcal{F}$ continuous.

Proposition 3. *Let x_n be a sequence in X and let $x \in X$. Then, the following are equivalent:*

(1) *$x_n \to x$ in the topology of $\sigma(X, \mathcal{F})$;*

(2) *$f(x_n) \to f(x)$, for every $f \in \mathcal{F}$.*

Exercise 4. *Let Z be a topological space and let $\phi : Z \to X$. Prove that the following are equivalent:*

(1) *ϕ is continuous with respect to the topology $\sigma(X, \mathcal{F})$;*

(2) *$f(\varphi) : Z \to \mathbb{R}$ is continuous for every $f \in \mathcal{F}$.*

The weak topology on a Banach space $\mathcal{B}$

Let $\mathcal{B}$ be a Banach space and let $\mathcal{B}'$ be its dual.

Definition 5 (Weak topology)**.** *The weak topology on $\mathcal{B}$ is the topology $\sigma(\mathcal{B}, \mathcal{B}')$, which is:*

the weakest topology on $\mathcal{B}$ that makes all the maps in $\mathcal{B}'$ continuous.

Remark 6 (Strong topology)**.** *The strong topology on $\mathcal{B}$ is the topology induced by the norm $\|\cdot\|_{\mathcal{B}}$.*

Remark 7 (Weak topology and weak convergence)**.** *Let x_n be a sequence in $\mathcal{B}$ and let $x \in \mathcal{B}$. Then, the following are equivalent:*

(1) *$x_n \to x$ in the topology of $\sigma(\mathcal{B}, \mathcal{B}')$;*

(2) *$T(x_n) \to T(x)$, for every $T \in \mathcal{B}'$.*

We will say that x_n converges to x weakly in $\mathcal{B}$ and we will write $x_n \rightharpoonup x$.

Proposition 8 (The weak topology is Hausdorff)**.** *Let $\mathcal{B}$ be a Banach space and let $\sigma(\mathcal{B}, \mathcal{B}')$ be the weak topology on $\mathcal{B}$. Then, for every couple of distinct points $x, y \in \mathcal{B}$, there are open sets $V, W \in \sigma(\mathcal{B}, \mathcal{B}')$ such that:*

$$x \in V \ , \quad y \in W \ , \quad V \cap W = \emptyset.$$

Proof. The claim follows from the Hahn-Banach's theorems. We consider three cases.
Case 1. Suppose that one between x and y is zero, say $x = 0$ and $y \neq 0$. By Hahn-Banach, there is $T \in \mathcal{B}'$ such that:

$$T(0) = 1 \quad \text{and} \quad T(y) = 1 > 0.$$

Then,

$$x = 0 \in \left\{T < \frac{1}{2}\right\} \qquad \text{and} \qquad y \in \left\{T > \frac{1}{2}\right\}.$$

Case 2. Suppose that $x \neq 0$ and $y = tx$ for some $t > 1$. By Hahn-Banach, there is $T \in \mathcal{B}'$ such that:

$$T(x) = 1 \quad \text{and} \quad T(y) = t > 0.$$

Then,

$$x \in \left\{T < \frac{1+t}{2}\right\} \qquad \text{and} \qquad y \in \left\{T > \frac{1+t}{2}\right\}.$$

Case 3. Suppose that $y \notin \left\{tx \ : \ t \in \mathbb{R}\right\}$. By Hahn-Banach, there is $T \in \mathcal{B}'$ such that:

$$T(x) = 0 \quad \text{and} \quad T(y) = 1.$$

Then,

$$x \in \left\{T < \frac{1}{2}\right\} \qquad \text{and} \qquad x \in \left\{T > \frac{1}{2}\right\}.$$

$\square$

Proposition 9 (Boundedness and semicontinuity of the norm)**.** *Let x_n be a sequence in $\mathcal{B}$ and let $x \in \mathcal{B}$. If $x_n \rightharpoonup x$, then the sequence of norms $\|x_n\|_{\mathcal{B}}$ is bounded and*

$$\|x\|_{\mathcal{B}} \leq \liminf_{n \to +\infty} \|x_n\|_{\mathcal{B}}.$$

Proof. The sequence of norms is bounded thanks to Banach-Steinhaus. In order to prove the lower-semicontinuity of the norms, we consider an operator

$$T \in \mathcal{B}' , \qquad T(x) = \|x\|_{\mathcal{B}} , \qquad \|T\|_{\mathcal{B}'} = 1,$$

which exists thanks to Hahn-Banach. Now, take a subsequence x_{n_k} such that

$$\liminf_{k\to+\infty} \|x_{n_k}\|_{\mathcal{B}} = \liminf_{n\to+\infty} \|x_n\|_{\mathcal{B}},$$

and compute

$$\|x\|_{\mathcal{B}} = T(x) = \lim_{n\to+\infty} T(x_{n_k}) \le \lim_{k\to+\infty} \|T\|_{\mathcal{B}'}\|x_{n_k}\|_{\mathcal{B}} = \liminf_{n\to+\infty} \|x_n\|_{\mathcal{B}}.$$

□

Proposition 10. *Let x_n be a sequence in $\mathcal{B}$ and let $x \in \mathcal{B}$. Let T_n be a sequence in $\mathcal{B}'$ and let $T \in \mathcal{B}'$. If*

$$x_n \rightharpoonup x \qquad \text{and} \qquad T_n \to T,$$

then

$$T(x) = \lim_{n\to+\infty} T_n(x_n).$$

Proof. We write

$$\begin{aligned} |T_n(x_n) - T(x)| &= |T_n(x_n) - T(x_n)| + |T(x_n) - T(x)| \\ &\le \|T_n - T\|_{\mathcal{B}'}\|x_n\|_{\mathcal{B}} + |T(x_n) - T(x)|. \end{aligned}$$

By the strong convergence of T_n to T, we have $\|T_n - T\|_{\mathcal{B}'} \to 0$, while the weak convergence of x_n gives that $\|x_n\|_{\mathcal{B}}$ is bounded. Thus,

$$\|T_n - T\|_{\mathcal{B}'}\|x_n\|_{\mathcal{B}} \to 0.$$

On the other hand, the weak convergence of x_n gives

$$|T(x_n) - T(x)| \to 0,$$

which concludes the proof. □

Comparison of the weak and the strong topologies on $\mathcal{B}$

Proposition 11. *Suppose that $\mathcal{B}$ is a finite dimensional Banach space, that is, $\mathcal{B} = \mathbb{R}^N$. Then, the weak and the strong topologies on $\mathcal{B}$ are equivalent.*

Proposition 12. *Suppose that $\mathcal{B}$ is an infinite dimensional Banach space. Let S_1 be the unit sphere*

$$S_1 = \Big\{x \in \mathcal{B} \ : \ \|x\|_{\mathcal{B}} = 1\Big\}.$$

Then, the closure of S_1 with respect to the weak topology $\sigma(\mathcal{B}, \mathcal{B}')$ is the closed unit ball

$$\overline{B}_1 = \Big\{x \in \mathcal{B} \ : \ \|x\|_{\mathcal{B}} \le 1\Big\}.$$

Proof. Let $x_0 \in B_1$. It is sufficient to prove that any neighborhood $V \in \mathcal{N}(\mathcal{B}, \mathcal{B}')$ defined as

$$V = \bigcap_{i=1}^{N} T_i^{-1}((-\varepsilon_i + T_i(x_0), T_i(x_0) + \varepsilon_i)),$$

contains a point on the sphere. Notice that since $\mathcal{B}$ is finite dimensional the map

$$L : \mathcal{B} \to \mathbb{R}^N , \qquad L(x) = (T_1(x), \dots, T_N(x)),$$

cannot be injective. Then, there is $y \in \mathcal{B}$, $y \neq 0$, such that

$$T_1(y) = \cdots = T_N(y) = 0.$$

But then, there is $t \in \mathbb{R}$ such that

$$\|x_0 + ty\| = 1 \qquad \text{and} \qquad x_0 + ty \in V.$$

$\square$

Corollary 13. *Suppose that $\mathcal{B}$ is an infinite dimensional Banach space. Then, the unit ball*

$$B_1 = \Big\{x \in \mathcal{B} \ : \ \|x\|_{\mathcal{B}} < 1\Big\},$$

is not an open set with respect to the weak topology $\sigma(\mathcal{B}, \mathcal{B}')$. In particular, the weak and the strong topologies on $\mathcal{B}$ are not the same.

Proof. It is sufficient to notice that the set $\Big\{x \in \mathcal{B} \ : \ \|x\|_{\mathcal{B}} \geq 1\Big\}$ is closed with respect to the strong topology, but its closure with respect to $\sigma(\mathcal{B}, \mathcal{B}')$ is th whole space $\mathcal{B}$. $\square$

The weak topology on $\mathcal{B}$ is not induced by a metric

Lemma 14. *Let V be a vector space and let*

$$T_i : V \to \mathbb{R} \ , \qquad i = 1, \dots, N,$$

be linear maps. Suppose that $T : V \to \mathbb{R}$ is a linear map with satisfying the following property:

if $v \in V$ is such that $T_i(v) = 0$ for every $i = 1, \dots, N$, then $T(v) = 0$.

Then, T is a linear combination of T_i.

Proof. Consider the linear map

$$L : V \to \mathbb{R}^{N+1} \ , \quad L(v) = \Big(T(v), T_1(v), \dots, T_N(v)\Big).$$

Then, $L(V)$ is a linear subspace of $\mathbb{R}^{N+1}$ and $e_1 = (1, 0, \dots, 0) \notin L(V)$. Then, there is a linear functional

$$\xi : \mathbb{R}^{N+1} \to \mathbb{R} \ , \qquad \xi(x) = \lambda x_1 + \sum_{j=1}^{N} \lambda_j x_{1+j}.$$

such that

$$\xi(e_1) = 1 \quad \text{and} \quad \xi \equiv 0 \text{ on } L(V).$$

Since $\xi(T(v)) \equiv 0$ we have

$$0 = \lambda T(v) + \sum_{j=1}^{N} \lambda_j T_j(v).$$

On the other hand, by construction we have $\lambda \neq 0$, so

$$T(v) = -\frac{1}{\lambda} \sum_{j=1}^{N} \lambda_j T_j(v) \quad \text{for every} \quad v \in V.$$

$\square$

Lemma 15. *Let $\mathcal{B}$ be a Banach space. If $\mathcal{B}'$ is finite dimensional, then also $\mathcal{B}$ is finite dimensional.*

Proof. Suppose that T_k, $k = 1, \ldots, N$ is a basis of $\mathcal{B}'$. Consider the map

$$L : \mathcal{B} \to \mathbb{R}^N \ , \quad L(v) = \Big(T_1(v), \ldots, T_N(v)\Big).$$

First we notice that by the Hahn-Banach's theorem the kernel of L contains only the origin. Then L is injective. $\square$

Lemma 16. *Let $\mathcal{B}$ be a Banach space. Suppose that there is a countable family of vectors v_n that spans $\mathcal{B}$. Then, $\mathcal{B}$ is finite dimensional.*

Proof. Follows from the Baire's lemma. $\square$

Theorem 17. *Let $\mathcal{B}$ be an infinite dimensional Banach space. Then, the weak topology $\sigma(\mathcal{B}, \mathcal{B}')$ is NOT induced by a norm or a metric on $\mathcal{B}$.*

Proof. Suppose by contradiction that there is a distance δ on $\mathcal{B}$. Let now $T \in \mathcal{B}'$. Then, the set

$$W = \{x \in \mathcal{B} \ : \ |T(x)| < 1\}$$

is an open neighborhood of 0. Since T is continuous, there is a metric ball

$$B_r^\delta = \{x \in \mathcal{V} \ : \ \delta(x, 0) < r\}.$$

Since $\sigma(\mathcal{B}, \mathcal{B}')$ is induced by δ, there is a neighborhood $V \in \mathcal{N}(\mathcal{B}, \mathcal{B}')$ defined as

$$V = \bigcap_{i=1}^N T_i^{-1}((-\varepsilon_i + T_i(x_0), T_i(x_0) + \varepsilon_i)).$$

such that $V \subset B_r^\delta$. Suppose now that $y \in \mathcal{B}$ satisfies

$$T_1(y) = \ldots T_N(y) = 0.$$

Then, for all $t \in \mathbb{R}$,

$$T_1(ty) = \ldots T_N(ty) = 0.$$

Thus $ty \in V$ for every $t > 0$. By construction

$$ty \in \{x \in \mathcal{B} \ : \ |T(x)| < 1\}$$

for every $t \in \mathbb{R}$. But then $T(y) = 0$. By the previous lemma, T is of the form:

$$T = \alpha_1 T_1 + \alpha_2 T_2 + \cdots + \alpha_N T_N.$$

This means that $\mathcal{B}'$ is generated by a countable family of operators T_k, $k \in \mathbb{N}$. But then, $\mathcal{B}'$ has to be finite dimensional. But then also $\mathcal{B}$ has to be finite dimensional. $\square$

The weak topology on the unit ball of $\mathcal{B}$ can be induced by a metric

Proposition 18. *Let $\mathcal{B}$ be a Banach space with separable dual $\mathcal{B}'$. Then, the weak topology $\sigma(\mathcal{B}, \mathcal{B}')$ on the unit ball*

$$\overline{B}_1 = \Big\{x \in \mathcal{B} \ : \ \|x\|_{\mathcal{B}} \leq 1\Big\},$$

is metrizable.

Proof. Consider a countable dense subset $(T_n)_{n\geq 1}$ of $\mathcal{B}'$ and set

$$\widetilde{T}_n := \frac{1}{\|T_n\|_{\mathcal{B}'}} T_n \in \mathcal{B}'.$$

Define

$$\delta(x,y) := \sum_{n=1}^{+\infty} \frac{1}{2^n} |\widetilde{T}_n(x-y)|.$$

Step 1. δ is a metric on $\mathcal{B}$. Indeed, it is immediate to check that since

$$|\widetilde{T}_n(x-z)| \leq |\widetilde{T}_n(x-y)| + |\widetilde{T}_n(y-z)|,$$

the triangular inequality

$$\delta(x,z) \leq \delta(x,y) + \delta(y,z),$$

holds for all $x, y, z \in \mathcal{B}$. Moreover, if $\delta(x,y) = 0$, then

$$\widetilde{T}_n(x-y) = 0 \quad \text{for all} \quad n \geq 1.$$

Suppose now that $x - y \neq 0$. By the Hahn-Banach's theorem, there is $T \in \mathcal{B}$ such that

$$T(x-y) = \|x-y\|_{\mathcal{B}} \quad \text{and} \quad \|T\|_{\mathcal{B}'} = 1.$$

But, the density of $(T_n)_{n\geq 1}$ implies that there is a subsequence $(T_{n_k})_{k\geq 1}$ such that

$$T_{n_k} \to T \quad \text{strongly in } \mathcal{B}',$$

which also implies that

$$\widetilde{T}_{n_k} \to T \quad \text{strongly in } \mathcal{B}'.$$

Then,

$$\widetilde{T}_{n_k}(x-y) \to T(x-y),$$

and so, for large enough k, $\widetilde{T}_{n_k}(x-y) \neq 0$, which is a contradiction.

Step 2. The topology of δ is equivalent to $\sigma(\mathcal{B}, \mathcal{B}')$. Suppose that x_n is a sequence in $\overline{B}_1$ converging weakly to some $x \in \overline{B}_1$. Then:

- $|\widetilde{T}_n(x_k - x)| \leq 2$ for all $n \geq 1$ and all $k \geq 1$;
- for all fixed $n \geq 1$, $|\widetilde{T}_n(x_k - x)| \to 0$ as $k \to +\infty$.

As a consequence

$$\lim_{k\to+\infty} \delta(x_k, x) = 0.$$

Conversely, suppose that

$$\lim_{k\to+\infty} \delta(x_k, x) = 0.$$

Then, for every fixed $n \geq 1$ we have

$$\lim_{k\to+\infty} |\widetilde{T}_n(x_k - x)| = 0,$$

so also

$$\lim_{k\to+\infty} |T_n(x_k - x)| = 0.$$

Since the sequence $(x_k - x)$ is bounded in $\mathcal{B}$, we have that $x_k \rightharpoonup x$. $\square$