Liminf e limsup. Esercizi.

Esercizi

Esercizio 1. *Studiare il comportamento (esistenza di limite, limsup e liminf), per $(x,y) \to (0,0)$, delle funzioni seguenti:*

(1) $F(x,y) = \dfrac{x^2+y^2}{x^2+y^2-xy}$;

(2) $F(x,y) = \dfrac{x^2-xy^2-y^2}{x^2+y^2}$;

(3) $F(x,y) = \dfrac{x^3+x^2y^2}{x^2+y^2}$;

(4) $F(x,y) = \dfrac{e^x-1}{\sqrt{x^2+y^2}}$;

(5) $F(x,y) = \dfrac{e^{x+y}-1}{\sqrt{x^2+y^2}}$;

(6) $F(x,y) = \dfrac{e^x-e^y}{\sqrt{x^2+y^2}}$;

(7) $F(x,y) = \dfrac{\cos(\sqrt{x^2+y^2})-\cos x}{x^2+y^2}$;

(8) $F(x,y) = \dfrac{\cos(x)-\cos(y)}{\sqrt{x^2+y^2}}$;

(9) $F(x,y) = \dfrac{(e^y-1)\sin x}{(x^2+y^2)^{3/2}}$.

Esercizio 2. *Usando le coordinate polari, calcolare limsup e liminf, per $(x,y) \to (0,0)$, della funzione*

$$F(x,y) := \frac{xy^2}{x^2+y^4} .$$

Esercizio 3. *Data la funzione $F : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$*

$$F(x,y) = \frac{x^2y^2}{\sqrt{x^2+y^2}(x^2+y^4)},$$

calcolare

$$\limsup_{(x,y)\to(0,0)} F(x,y) \qquad e \qquad \liminf_{(x,y)\to(0,0)} F(x,y).$$

Esercizio 4. *Data la funzione $F : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$*

$$F(x,y) = \frac{xy^3}{\sqrt{x^2+y^2}(x^2+y^4)},$$

calcolare

$$\limsup_{(x,y)\to(0,0)} F(x,y) \qquad e \qquad \liminf_{(x,y)\to(0,0)} F(x,y).$$

Soluzione di Esercizio 2

Osservazione 5. *Osserviamo che*

$$\limsup_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} \qquad e \qquad \liminf_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

si possono calcolare anche senza l'utilizzo delle coordinate polari. Infatti,

$$-\frac{1}{2} \le \frac{xy^2}{x^2+y^4} \le \frac{1}{2} \qquad \textit{per ogni} \qquad (x,y) \neq (0,0),$$

con uguaglianze raggiunte quando $x = y^2$ e $x = -y^2$. Quindi

$$\limsup_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \frac{1}{2} \qquad e \qquad \liminf_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = -\frac{1}{2}\,.$$

Soluzione di Esercizio 2. In coordinate polari abbiamo

$$F\big(r\cos\theta, r\sin\theta\big) = \frac{r^3\cos\theta\sin^2\theta}{r^2\cos^2\theta + r^4\sin^4\theta} = \frac{r\cos\theta\sin^2\theta}{\cos^2\theta + r^2\sin^4\theta}\,.$$

Quindi

$$\partial_\theta\Big[F\big(r\cos\theta, r\sin\theta\big)\Big] = 0$$

se e solo se

$$\begin{aligned}
0 &= \partial_\theta\Big[r\cos\theta\sin^2\theta\Big]\Big(\cos^2\theta + r^2\sin^4\theta\Big) - r\cos\theta\sin^2\theta\,\partial_\theta\Big[\cos^2\theta + r^2\sin^4\theta\Big] \\
&= \Big(2r\sin\theta\cos^2\theta - r\sin^3\theta\Big)\Big(\cos^2\theta + r^2\sin^4\theta\Big) - r\cos\theta\sin^2\theta\Big(-2\sin\theta\cos\theta + 4r^2\sin^3\theta\cos\theta\Big) \\
&= r\sin\theta\Big(2\cos^2\theta - \sin^2\theta\Big)\Big(\cos^2\theta + r^2\sin^4\theta\Big) + r\sin\theta\Big(2\sin^2\theta\cos^2\theta - 4r^2\sin^4\theta\cos^2\theta\Big) \\
&= r\sin\theta\Big[\cos^2\theta\Big(2 - \sin^2\theta\Big) + r^2\Big(\sin^4\theta\Big(2\cos^2\theta - \sin^2\theta\Big) - 4\sin^4\theta\cos^2\theta\Big)\Big] \\
&= r\sin\theta\Big[\cos^2\theta\Big(2 - \sin^2\theta\Big) - r^2\Big(\sin^6\theta + 2\sin^4\theta\cos^2\theta\Big)\Big] \\
&= r\sin\theta\Big[\cos^2\theta\Big(2 - \sin^2\theta\Big) - r^2\sin^4\theta\Big(1 + \cos^2\theta\Big)\Big].
\end{aligned}$$

Quando $\sin\theta = 0$, abbiamo che $F\big(r\cos\theta, r\sin\theta\big) = 0$.

Consideriamo il caso

$$\cos^2\theta = \frac{r^2\sin^4\theta(1+\cos^2\theta)}{2-\sin^2\theta}.$$

Allora,

$$\cos^2\theta \le 2r^2,$$

e di conseguenza

$$\sin^2\theta = 1 - \cos^2\theta = 1 + O(r^2).$$

Tornando all'equazione per θ,

$$\cos^2\theta = \frac{r^2\big(1+O(r^2)\big)\big(1+O(r^2)\big)}{2 - \big(1+O(r^2)\big)} = r^2\big(1+O(r^2)\big).$$

Quindi,

$$\sup_\theta F\big(r\cos\theta, r\sin\theta\big) = \frac{r^2\sqrt{1+O(r^2)}\,\big(1+O(r^2)\big)}{r^2\big(1+O(r^2)\big) + r^2\big(1+O(r^2)\big)^2} = \frac{1}{2} + O(r^2)\,;$$

$$\inf_\theta F\big(r\cos\theta, r\sin\theta\big) = -\frac{r^2\sqrt{1+O(r^2)}\,\big(1+O(r^2)\big)}{r^2\big(1+O(r^2)\big)+r^2\big(1+O(r^2)\big)^2} = -\frac{1}{2}+O(r^2)\,.$$

Di conseguenza,

$$\limsup_{r\to 0}\sup_\theta F\big(r\cos\theta, r\sin\theta\big) = \frac{1}{2} \qquad \text{e} \qquad \liminf_{r\to 0}\inf_\theta F\big(r\cos\theta, r\sin\theta\big) = -\frac{1}{2}$$

□

Soluzione di Esercizio 3

In coordinate polari abbiamo

$$F(r\cos\theta, r\sin\theta) = \frac{r^4\cos^2\theta\sin^2\theta}{r(r^2\cos^2\theta + r^4\sin^4\theta)} = \frac{r\cos^2\theta\sin^2\theta}{\cos^2\theta + r^2\sin^4\theta}.$$

Quindi

$$\partial_\theta\left[\frac{r\cos^2\theta\sin^2\theta}{\cos^2\theta + r^2\sin^4\theta}\right] = 0$$

se e solo se

$$\begin{aligned}
0 &= \partial_\theta\Big[r\cos^2\theta\sin^2\theta\Big]\Big(\cos^2\theta + r^2\sin^4\theta\Big) - \Big(r\cos^2\theta\sin^2\theta\Big)\partial_\theta\Big[\cos^2\theta + r^2\sin^4\theta\Big]\\
&= \Big[-2r\sin^3\theta\cos\theta + 2r\cos^3\theta\sin\theta\Big]\Big(\cos^2\theta + r^2\sin^4\theta\Big) - \Big(r\cos^2\theta\sin^2\theta\Big)\Big[-2\sin\theta\cos\theta + 4r^2\cos\theta\sin^3\theta\Big]\\
&= 2r\sin\theta\cos\theta\Big[\Big(-\sin^2\theta + \cos^2\theta\Big)\Big(\cos^2\theta + r^2\sin^4\theta\Big) + \cos^2\theta\sin^2\theta\Big(1 - 2r^2\sin^2\theta\Big)\Big]\\
&= 2r\sin\theta\cos\theta\Big(\cos^4\theta - r^2\sin^4\theta\Big)\\
&= 2r\sin\theta\cos\theta\Big(\cos^2\theta + r\sin^2\theta\Big)\Big(\cos^2\theta - r\sin^2\theta\Big).
\end{aligned}$$

Quindi le soluzioni sono

$$\cos\theta = 0,\ \sin\theta = 0, \quad \text{oppure} \quad \cos^2\theta = r\sin^2\theta,$$

che (siccome $\cos^2\theta + \sin^2\theta = 1$) possiamo scrivere anche come

$$\cos\theta = 0,\ \sin\theta = 0, \quad \text{oppure} \quad \begin{cases}\cos^2\theta = \frac{r}{1+r}\\ \sin^2\theta = \frac{1}{1+r}\end{cases}.$$

Caso 1.

$$\cos\theta = 0 \quad \text{oppure} \quad \sin\theta = 0.$$

In questo caso

$$F(r\cos\theta, r\sin\theta) = 0.$$

Caso 2.

$$\begin{cases}\cos^2\theta = \frac{r}{1+r}\\ \sin^2\theta = \frac{1}{1+r}\end{cases}.$$

Allora,

$$F(r\cos\theta, r\sin\theta) = \frac{r\left(\frac{1}{1+r}\right)\left(\frac{r}{1+r}\right)}{\left(\frac{r}{1+r}\right) + r^2\left(\frac{1}{1+r}\right)^2} = \frac{r^2}{r(1+r)+r^2} = \frac{r}{1+2r}.$$

Di conseguenza,

$$\max_{\theta\in[0,2\pi]} F(r\cos\theta, r\sin\theta) = \frac{r}{1+2r} \qquad \text{e} \qquad \max_{\theta\in[0,2\pi]} F(r\cos\theta, r\sin\theta) = 0.$$

In conclusione,

$$\limsup_{(x,y)\to(0,0)} F(x,y) = \liminf_{(x,y)\to(0,0)} F(x,y) = 0.$$

Soluzione di Esercizio 4

In coordinate polari abbiamo

$$F(r\cos\theta, r\sin\theta) = \frac{r^4\cos\theta\sin^3\theta}{r(r^2\cos^2\theta + r^4\sin^4\theta)} = \frac{r\cos\theta\sin^3\theta}{\cos^2\theta + r^2\sin^4\theta}.$$

Quindi

$$\partial_\theta\left[\frac{r\cos\theta\sin^3\theta}{\cos^2\theta + r^2\sin^4\theta}\right] = 0$$

se e solo se

$$\begin{aligned}
0 &= \partial_\theta\Big[r\cos\theta\sin^3\theta\Big]\Big(\cos^2\theta + r^2\sin^4\theta\Big) - \Big(r\cos\theta\sin^3\theta\Big)\partial_\theta\Big[\cos^2\theta + r^2\sin^4\theta\Big]\\
&= \Big[-r\sin^4\theta + 3r\cos^2\theta\sin^2\theta\Big]\Big(\cos^2\theta + r^2\sin^4\theta\Big) - \Big(r\cos\theta\sin^3\theta\Big)\Big[-2\sin\theta\cos\theta + 4r^2\cos\theta\sin^3\theta\Big]\\
&= r\sin^2\theta\Big[\Big(-\sin^2\theta + 3\cos^2\theta\Big)\Big(\cos^2\theta + r^2\sin^4\theta\Big) + \cos^2\theta\sin^2\theta\Big(2 - 4r^2\sin^2\theta\Big)\Big]\\
&= r\sin^2\theta\Big(\cos^2\theta(1+2\cos^2\theta) - r^2\sin^4\theta\Big).
\end{aligned}$$

Quindi le soluzioni sono

$$\sin\theta = 0, \quad \text{oppure} \quad \cos^2\theta(1+2\cos^2\theta) = r^2(1-\cos^2\theta)^2.$$

Caso 1. $\sin\theta = 0$. In questo caso

$$F(r\cos\theta, r\sin\theta) = 0.$$

Caso 2. $\cos^2\theta(1+2\cos^2\theta) = r^2(1-\cos^2\theta)^2$. Osserviamo che se θ è soluzione di

$$\cos^2\theta(1+2\cos^2\theta) = r^2(1-\cos^2\theta)^2,$$

allora necessariamente $\cos^2\theta \le r^2$ e di conseguenza

$$r^2 = \cos^2\theta\frac{(1+2\cos^2\theta)}{(1-\cos^2\theta)^2} \le \cos^2\theta\frac{1+2r^2}{(1-r^2)^2},$$

e quindi

$$r^2\frac{(1-r^2)^2}{1+2r^2} \le \cos^2\theta \le r^2.$$

Possiamo scrivere quindi

$$\cos^2\theta = r^2(1+O(r^2)) \qquad \text{e} \qquad \sin^2\theta = 1 + O(r^2).$$

Di conseguenza, se $\cos\theta\sin\theta$ è positivo, allora

$$F(r\cos\theta, r\sin\theta) = \frac{r\cos\theta\sin^3\theta}{\cos^2\theta + r^2\sin^4\theta} = \frac{r^2(1+O(r^2))^{1/2}(1+O(r^2))^{3/2}}{r^2(1+O(r^2)) + r^2(1+O(r^2))^2} = \frac{1}{2}(1+O(r^2)).$$

Di conseguenza,

$$\lim_{r\to0}\Big\{\max_{\theta\in[0,2\pi]} F(r\cos\theta, r\sin\theta)\Big\} = \frac{1}{2}.$$

Analogamente,

$$\lim_{r\to0}\Big\{\min_{\theta\in[0,2\pi]} F(r\cos\theta, r\sin\theta)\Big\} = -\frac{1}{2}.$$

In conclusione,

$$\limsup_{(x,y)\to(0,0)} F(x,y) = \frac{1}{2} \qquad \text{e} \qquad \liminf_{(x,y)\to(0,0)} F(x,y) = -\frac{1}{2}.$$