

UN ALTRO ESERCIZIO SUL CALCOLO DI \limsup E \liminf ALL'INFINITO

Esercizio 1. Data la funzione $F: \mathbb{R}^2 \rightarrow \mathbb{R}$

$$F(x, y) = \frac{xy + x}{x^2 + y^2 + 1},$$

calcolare $\limsup_{|(x,y)| \rightarrow +\infty} F(x, y)$ e $\liminf_{|(x,y)| \rightarrow +\infty} F(x, y)$.

Soluzione. In coordinate polari abbiamo

$$F(R \cos \theta, R \sin \theta) = \frac{R^2 \cos \theta \sin \theta + R \cos \theta}{R^2 + 1}.$$

Ora scriviamo

$$F(R \cos \theta, R \sin \theta) = f(R, \theta) + g(R, \theta),$$

dove

$$f(R, \theta) = \frac{R^2 \cos \theta \sin \theta}{R^2 + 1} \quad \text{e} \quad g(R, \theta) = \frac{R \cos \theta}{R^2 + 1}.$$

Abbiamo che

$$|g(R, \theta)| \leq \frac{R}{1 + R^2} \quad \text{per ogni} \quad \theta \in [0, 2\pi].$$

Di conseguenza,

$$\sup_{\theta \in [0, 2\pi]} f(R, \theta) - \frac{R}{1 + R^2} \leq \sup_{\theta \in [0, 2\pi]} F(R \cos \theta, R \sin \theta) \leq \sup_{\theta \in [0, 2\pi]} f(R, \theta) + \frac{R}{1 + R^2}.$$

$$\inf_{\theta \in [0, 2\pi]} f(R, \theta) - \frac{R}{1 + R^2} \leq \inf_{\theta \in [0, 2\pi]} F(R \cos \theta, R \sin \theta) \leq \inf_{\theta \in [0, 2\pi]} f(R, \theta) + \frac{R}{1 + R^2}.$$

Passando al limite per $R \rightarrow +\infty$, otteniamo che

$$\limsup_{R \rightarrow +\infty} \left\{ \sup_{\theta \in [0, 2\pi]} F(R \cos \theta, R \sin \theta) \right\} = \limsup_{R \rightarrow +\infty} \left\{ \sup_{\theta \in [0, 2\pi]} f(R, \theta) \right\},$$

$$\liminf_{R \rightarrow +\infty} \left\{ \inf_{\theta \in [0, 2\pi]} F(R \cos \theta, R \sin \theta) \right\} = \liminf_{R \rightarrow +\infty} \left\{ \inf_{\theta \in [0, 2\pi]} f(R, \theta) \right\}.$$

Quindi, basta calcolare

$$\limsup_{R \rightarrow +\infty} \left\{ \sup_{\theta \in [0, 2\pi]} f(R, \theta) \right\} \quad \text{e} \quad \liminf_{R \rightarrow +\infty} \left\{ \inf_{\theta \in [0, 2\pi]} f(R, \theta) \right\}.$$

Osserviamo che

$$f(r, \theta) = \frac{R^2 \cos \theta \sin \theta}{R^2 + 1} = \frac{R^2}{R^2 + 1} \frac{\sin(2\theta)}{2},$$

e, siccome $\sup_{\theta \in [0, 2\pi]} \sin(2\theta) = 1$ e $\inf_{\theta \in [0, 2\pi]} \sin(2\theta) = -1$,

$$\sup_{\theta \in [0, 2\pi]} f(R, \theta) = \frac{1}{2} \frac{R^2}{R^2 + 1} \quad \text{e} \quad \inf_{\theta \in [0, 2\pi]} f(R, \theta) = -\frac{1}{2} \frac{R^2}{R^2 + 1}.$$

Passando al limite per $R \rightarrow +\infty$,

$$\limsup_{R \rightarrow +\infty} \left\{ \sup_{\theta \in [0, 2\pi]} f(R, \theta) \right\} = \frac{1}{2} \quad \text{e} \quad \liminf_{R \rightarrow +\infty} \left\{ \inf_{\theta \in [0, 2\pi]} f(R, \theta) \right\} = -\frac{1}{2}.$$

In conclusione

$$\limsup_{|(x,y)| \rightarrow +\infty} F(x, y) = \frac{1}{2} \quad \text{e} \quad \liminf_{|(x,y)| \rightarrow +\infty} F(x, y) = -\frac{1}{2}.$$

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