

DUE ESERCIZI SUL CALCOLO DI \limsup E \liminf

Esercizio 1. Data la funzione $F : \mathbb{R}^2 \setminus \{(0,0)\} \rightarrow \mathbb{R}$

$$F(x, y) = \frac{x^2 y^2}{\sqrt{x^2 + y^2}(x^2 + y^4)},$$

calcolare

$$\limsup_{(x,y) \rightarrow (0,0)} F(x, y) \quad e \quad \liminf_{(x,y) \rightarrow (0,0)} F(x, y).$$

Dimostrazione. In coordinate polari abbiamo

$$F(r \cos \theta, r \sin \theta) = \frac{r^4 \cos^2 \theta \sin^2 \theta}{r(r^2 \cos^2 \theta + r^4 \sin^4 \theta)} = \frac{r \cos^2 \theta \sin^2 \theta}{\cos^2 \theta + r^2 \sin^4 \theta}.$$

Quindi

$$\partial_\theta \left[\frac{r \cos^2 \theta \sin^2 \theta}{\cos^2 \theta + r^2 \sin^4 \theta} \right] = 0$$

se e solo se

$$\begin{aligned} 0 &= \partial_\theta \left[r \cos^2 \theta \sin^2 \theta \right] \left(\cos^2 \theta + r^2 \sin^4 \theta \right) - \left(r \cos^2 \theta \sin^2 \theta \right) \partial_\theta \left[\cos^2 \theta + r^2 \sin^4 \theta \right] \\ &= \left[-2r \sin^3 \theta \cos \theta + 2r \cos^3 \theta \sin \theta \right] \left(\cos^2 \theta + r^2 \sin^4 \theta \right) - \left(r \cos^2 \theta \sin^2 \theta \right) \left[-2 \sin \theta \cos \theta + 4r^2 \cos \theta \sin^3 \theta \right] \\ &= 2r \sin \theta \cos \theta \left[\left(-\sin^2 \theta + \cos^2 \theta \right) \left(\cos^2 \theta + r^2 \sin^4 \theta \right) + \cos^2 \theta \sin^2 \theta \left(1 - 2r^2 \sin^2 \theta \right) \right] \\ &= 2r \sin \theta \cos \theta \left(\cos^4 \theta - r^2 \sin^4 \theta \right) \\ &= 2r \sin \theta \cos \theta \left(\cos^2 \theta + r \sin^2 \theta \right) \left(\cos^2 \theta - r \sin^2 \theta \right). \end{aligned}$$

Quindi le soluzioni sono

$$\cos \theta = 0, \sin \theta = 0, \quad \text{oppure} \quad \cos^2 \theta = r \sin^2 \theta,$$

che (siccome $\cos^2 \theta + \sin^2 \theta = 1$) possiamo scrivere anche come

$$\cos \theta = 0, \sin \theta = 0, \quad \text{oppure} \quad \begin{cases} \cos^2 \theta = \frac{r}{1+r} \\ \sin^2 \theta = \frac{1}{1+r} \end{cases}.$$

Caso 1.

$$\cos \theta = 0 \quad \text{oppure} \quad \sin \theta = 0.$$

In questo caso

$$F(r \cos \theta, r \sin \theta) = 0.$$

Caso 2.

$$\begin{cases} \cos^2 \theta = \frac{r}{1+r} \\ \sin^2 \theta = \frac{1}{1+r} \end{cases}.$$

Allora,

$$F(r \cos \theta, r \sin \theta) = \frac{r \left(\frac{1}{1+r} \right) \left(\frac{r}{1+r} \right)}{\left(\frac{r}{1+r} \right) + r^2 \left(\frac{1}{1+r} \right)^2} = \frac{r^2}{r(1+r) + r^2} = \frac{r}{1+2r}.$$

Di conseguenza,

$$\max_{\theta \in [0, 2\pi]} F(r \cos \theta, r \sin \theta) = \frac{r}{1+2r} \quad e \quad \max_{\theta \in [0, 2\pi]} F(r \cos \theta, r \sin \theta) = 0.$$

In conclusione,

$$\limsup_{(x,y) \rightarrow (0,0)} F(x, y) = \liminf_{(x,y) \rightarrow (0,0)} F(x, y) = 0.$$

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Esercizio 2. Data la funzione $F : \mathbb{R}^2 \setminus \{(0,0)\} \rightarrow \mathbb{R}$

$$F(x, y) = \frac{xy^3}{\sqrt{x^2 + y^2}(x^2 + y^4)},$$

calcolare

$$\limsup_{(x,y) \rightarrow (0,0)} F(x, y) \quad e \quad \liminf_{(x,y) \rightarrow (0,0)} F(x, y).$$

Dimostrazione. In coordinate polari abbiamo

$$F(r \cos \theta, r \sin \theta) = \frac{r^4 \cos \theta \sin^3 \theta}{r(r^2 \cos^2 \theta + r^4 \sin^4 \theta)} = \frac{r \cos \theta \sin^3 \theta}{\cos^2 \theta + r^2 \sin^4 \theta}.$$

Quindi

$$\partial_\theta \left[\frac{r \cos \theta \sin^3 \theta}{\cos^2 \theta + r^2 \sin^4 \theta} \right] = 0$$

se e solo se

$$\begin{aligned} 0 &= \partial_\theta \left[r \cos \theta \sin^3 \theta \right] \left(\cos^2 \theta + r^2 \sin^4 \theta \right) - \left(r \cos \theta \sin^3 \theta \right) \partial_\theta \left[\cos^2 \theta + r^2 \sin^4 \theta \right] \\ &= \left[-r \sin^4 \theta + 3r \cos^2 \theta \sin^2 \theta \right] \left(\cos^2 \theta + r^2 \sin^4 \theta \right) - \left(r \cos \theta \sin^3 \theta \right) \left[-2 \sin \theta \cos \theta + 4r^2 \cos \theta \sin^3 \theta \right] \\ &= r \sin^2 \theta \left[\left(-\sin^2 \theta + 3 \cos^2 \theta \right) \left(\cos^2 \theta + r^2 \sin^4 \theta \right) + \cos^2 \theta \sin^2 \theta \left(2 - 4r^2 \sin^2 \theta \right) \right] \\ &= r \sin^2 \theta \left(\cos^2 \theta (1 + 2 \cos^2 \theta) - r^2 \sin^4 \theta \right). \end{aligned}$$

Quindi le soluzioni sono

$$\sin \theta = 0, \quad \text{oppure} \quad \cos^2 \theta (1 + 2 \cos^2 \theta) = r^2 (1 - \cos^2 \theta)^2.$$

Caso 1. $\sin \theta = 0$. In questo caso

$$F(r \cos \theta, r \sin \theta) = 0.$$

Caso 2. $\cos^2 \theta (1 + 2 \cos^2 \theta) = r^2 (1 - \cos^2 \theta)^2$. Osserviamo che se θ è soluzione di

$$\cos^2 \theta (1 + 2 \cos^2 \theta) = r^2 (1 - \cos^2 \theta)^2,$$

allora necessariamente $\cos^2 \theta \leq r^2$ e di conseguenza

$$r^2 = \cos^2 \theta \frac{(1 + 2 \cos^2 \theta)}{(1 - \cos^2 \theta)^2} \leq \cos^2 \theta \frac{1 + 2r^2}{(1 - r^2)^2},$$

e quindi

$$r^2 \frac{(1 - r^2)^2}{1 + 2r^2} \leq \cos^2 \theta \leq r^2.$$

Possiamo scrivere quindi

$$\cos^2 \theta = r^2 (1 + O(r^2)) \quad e \quad \sin^2 \theta = 1 + O(r^2).$$

Di conseguenza, se $\cos \theta \sin \theta$ è positivo, allora

$$F(r \cos \theta, r \sin \theta) = \frac{r \cos \theta \sin^3 \theta}{\cos^2 \theta + r^2 \sin^4 \theta} = \frac{r^2 (1 + O(r^2))^{1/2} (1 + O(r^2))^{3/2}}{r^2 (1 + O(r^2)) + r^2 (1 + O(r^2))^2} = \frac{1}{2} (1 + O(r^2)).$$

Di conseguenza,

$$\lim_{r \rightarrow 0} \left\{ \max_{\theta \in [0, 2\pi]} F(r \cos \theta, r \sin \theta) \right\} = \frac{1}{2}.$$

Analogamente,

$$\lim_{r \rightarrow 0} \left\{ \min_{\theta \in [0, 2\pi]} F(r \cos \theta, r \sin \theta) \right\} = -\frac{1}{2}.$$

In conclusione,

$$\limsup_{(x,y) \rightarrow (0,0)} F(x, y) = \frac{1}{2} \quad e \quad \liminf_{(x,y) \rightarrow (0,0)} F(x, y) = -\frac{1}{2}.$$

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