Skew braces, brace blocks, and the Yang–Baxter equation

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Outline

- Skew braces and connections
- Some recent results

Skew braces and connections

Skew braces

Definition ([Guarnieri and Vendramin, 2017])

A *skew brace* is a triple $(G, \cdot, \circ)$, where $(G, \cdot)$ and $(G, \circ)$ are groups and for all $g, h, k \in G$,

$$g \circ (h \cdot k) = (g \circ h) \cdot g^{-1} \cdot (g \circ k).$$

(Here $^{-1}$ denotes the inverse with respect to $\cdot$.)

Example

Let $(G, \cdot)$ be a group.

- $(G, \cdot, \cdot)$ is a skew brace.
- $(G, \cdot, \circ)$ is a skew brace, where $g \circ h = h \cdot g$ for all $g, h \in G$.

Braces

Definition ([Rump, 2007], [Cedó et al., 2014])

A *brace* is a skew brace $(G, \cdot, \circ)$ such that $(G, \cdot)$ is abelian.

Example

For all $a, b \in (\mathbb{Z}, +)$, define $a \circ b = a + (-1)^a b$. Then $(\mathbb{Z}, +, \circ)$ is a brace.

The Yang–Baxter equation

Definition ([Drinfel'd, 1992])

A *(set-theoretic nondegenerate) solution* of the Yang–Baxter equation is a couple (X, r), where X is a nonempty set and

$$r\colon X \times X \to X \times X$$
$$(x, y) \mapsto (\sigma_x(y), \tau_y(x))$$

is a bijective map such that

$$(r \times \mathrm{id}_X)(\mathrm{id}_X \times r)(r \times \mathrm{id}_X) = (\mathrm{id}_X \times r)(r \times \mathrm{id}_X)(\mathrm{id}_X \times r)$$

and σ_x and τ_x are bijective for all $x \in X$.

We say that (X, r) is *involutive* if $r^2 = \mathrm{id}_{X\times X}$.

Theorem ([Rump, 2007],[Guarnieri and Vendramin, 2017])

Let $(G,\cdot,\circ)$ be a skew brace. Then (G,r) is a solution, where

$$r(g,h) = (g^{-1}\cdot(g\circ h), \overline{g^{-1}\cdot(g\circ h)}\circ g\circ h).$$

(Here an overline denotes the inverse with respect to $\circ$.)

The solution is involutive if and only if $(G,\cdot,\circ)$ is a brace.

Regular subgroups

Let $(G, 1)$ be a pointed set, and let $\mathrm{Perm}(G)$ be the group of permutations on G. A subgroup N of $\mathrm{Perm}(G)$ is *regular* if the map

$$\begin{aligned} N &\to G \\ \eta &\mapsto \eta[1] \end{aligned}$$

is a bijection.

If $(G, \circ)$ is a group with identity 1, then $\lambda_\circ(G)$ is regular, where

$$\begin{aligned} \lambda_\circ \colon G &\to \mathrm{Perm}(G) \\ g &\mapsto (h \mapsto g \circ h). \end{aligned}$$

Fact

Every regular subgroup arises in this way!

Skew braces and regular subgroups

Theorem ([Guarnieri and Vendramin, 2017])

Let $(G, \diamond)$ and $(G, \circ)$ be groups with identity 1. Then $(G, \diamond, \circ)$ is a skew brace if and only if $\lambda_\circ(G)$ normalises $\lambda_\diamond(G)$.

Let $(G, \cdot)$ be a group, and denote by $\mathrm{Hol}(G)$ the normaliser of $\lambda(G)$ in $\mathrm{Perm}(G)$.

Corollary

There is a bijective correspondence between operations $\circ$ such that $(G, \cdot, \circ)$ is a skew brace and regular subgroups of $\mathrm{Hol}(G)$.

Let L/K be a finite field extension.

Definition

A *Hopf–Galois structure* on L/K consists of the following data:

- a K-Hopf algebra H;
- an action of H on L such that certain technical properties are satisfied.

Example

If L/K is Galois with Galois group G, then $K[G]$ with the usual action is the *classical* Hopf–Galois structure on L/K.

Let L/K be a finite Galois extension with Galois group $(G, \cdot)$.

Theorem ([Greither and Pareigis, 1987])

There is a bijective correspondence between Hopf–Galois structures on L/K and regular subgroups N of $\mathrm{Perm}(G)$ normalised by $\lambda(G)$.

Corollary

There is a bijective correspondence between Hopf–Galois structures on L/K and operations $\circ$ on G such that $(G, \circ, \cdot)$ is a skew brace.

Bi-skew braces and brace blocks

Definition ([Childs, 2019])

We say that a skew brace $(G, \cdot, \circ)$ is a *bi-skew brace* if also $(G, \circ, \cdot)$ is a skew brace.

Definition ([Koch, 2022], [Caranti and LS, 2022a])

Let G be a set. A *brace block* on G is a family $(\circ_i \mid i \in I)$ of operations on G, where I is an index set, such that $(G, \circ_i, \circ_j)$ is a (bi-)skew brace for all $i, j \in I$.

Examples from [Koch, 2021, Koch, 2022]

Let $(G, \cdot)$ be a finite group, and let ψ be an abelian endomorphism of $(G, \cdot)$. Then $(G, \cdot, \circ)$ is a bi-skew brace, where for all $g, h \in G$,

$$g \circ h = g \cdot \psi(g)^{-1} \cdot h \cdot \psi(g).$$

As ψ is also an abelian endomorphism of $(G, \circ)$, the construction can be iterated, to obtain a brace block $(\circ_n \mid n \in \mathbb{N})$ on G, where $\circ_0 = \cdot$ and $\circ_1 = \circ$.

Some new results

A first generalisation

Let $(G, \cdot)$ be a group.

In [Caranti and LS, 2021], we characterised the endomorphisms ψ of G such that $(G, \cdot, \circ)$ is a bi-skew brace, where for all $g, h \in G$,

$$g \circ h = g \cdot \psi(g)^{\varepsilon} \cdot h \cdot \psi(g)^{-\varepsilon}.$$

(Here $\varepsilon = \pm 1$.)

For example, the result holds if $\psi([G, G]) \leq Z(G)$.

Question

- *Can we obtain a brace block from this construction?*
- *Do we have to assume that $\psi \in \mathrm{End}(G, \cdot)$?*

The setting

Let $(G, \cdot)$ be a group, let K be a subgroup of G contained in $Z(G)$, and let A be a subgroup of G such that A/K is abelian.

Consider the ring

$$\mathcal{A} = \{\psi \in \mathrm{End}(G/K) \mid \psi(G/K) \leq A/K\}.$$

For all $\psi \in \mathcal{A}$, define $\psi^{\uparrow}$ to be a lifting of ψ, that is, a set-theoretic map $\psi^{\uparrow}\colon G \to A$ such that

$$\psi^{\uparrow}(g)K = \psi(gK).$$

Note that for all $g, h \in G$,

$$\psi^{\uparrow}(g \cdot h) \equiv \psi^{\uparrow}(g) \cdot \psi^{\uparrow}(h) \pmod{K}.$$

The main theorem

Recall: $K \leq Z(G)$, A/K abelian,

$$\mathcal{A} = \{\psi \in \mathrm{End}(G/K) \mid \psi(G/K) \leq A/K\}.$$

For all $\psi \in \mathcal{A}$, define

$$g \circ_\psi h = g \cdot \psi^\uparrow(g) \cdot h \cdot (\psi^\uparrow(g))^{-1}.$$

Then $(G, \cdot, \circ_\psi)$ is a bi-skew brace.

Theorem ([Caranti and LS, 2022a])

The family $(\circ_\psi \mid \psi \in \mathcal{A})$ is a brace block on G.

Yang–Baxter in our setting

Let $(G, \cdot)$ be a group, and assume our main setting.

Theorem ([Caranti and LS, 2022a])

Let $\psi, \varphi \in \mathcal{A}$. Then (G, r) is a solution, where

$$\begin{aligned} r(g, h) = ((\psi - \varphi)^{\uparrow}(g) \cdot h \cdot ((\psi - \varphi)^{\uparrow}(g))^{-1}, \\ (\psi^{\uparrow}(h))^{-1} \cdot (\psi - \varphi)^{\uparrow}(g) \cdot h^{-1} \cdot ((\psi - \varphi)^{\uparrow}(g))^{-1} \\ \cdot g \cdot \psi^{\uparrow}(g) \cdot h \cdot (\psi^{\uparrow}(g))^{-1} \cdot \psi^{\uparrow}(h)). \end{aligned}$$

Groups of nilpotence class two

Let $(G, \cdot)$ be a group of nilpotence class two, let $K = [G, G]$, and let $A = G$. In this case $\mathcal{A} = \mathrm{End}(G/K)$.

- For all $n \in \mathbb{Z}$, take

$$g \circ_n h = g \cdot g^n \cdot h \cdot g^{-n} = g \cdot h \cdot [g, h]^n.$$

  Then $(\circ_n \mid n \in \mathbb{Z})$ is a brace block on G.

- For all $\psi \in \mathrm{End}(G)$, take

$$g \circ_\psi h = g \cdot \psi(g) \cdot h \cdot \psi(g)^{-1}.$$

  Then $(\circ_\psi \mid \psi \in \mathrm{End}(G))$ is a brace block on G.

The p-adic Heisenberg group

Let p be a prime number, and let $G = \{(a, b, c) \mid a, b, c \in \mathbb{Z}_p\}$ be the p-adic Heisenberg group, with group operation

$$(a, b, c) \cdot (a', b', c') = (a + a', b + b', c + c' + ab').$$

Then G is a topological group of nilpotence class two.

For all $x \in \mathbb{Z}_p$, define $\psi_x \colon G \to G$ by

$$\psi_x(a, b, c) = (xa, xb, x^2 c).$$

Then $\psi_x \in \mathrm{End}(G)$, and we obtain a brace block $(\circ_{\psi_x} \mid x \in \mathbb{Z}_p)$ on G.

Some pleasant facts

Explicitly, for all $g = (a, b, c), h = (a', b', c') \in G$,

$$g \circ_{\psi_x} h = g \cdot h \cdot (0, 0, x(ab' - a'b)).$$

In particular, the following facts hold:

- The brace block consists of infinitely many distinct operations.
- The skew braces of the kind $(G, \cdot, \circ_{\psi_z})$, $z \in \mathbb{Z}$, are not isomorphic.
- The operations $\circ_{\psi_{p^n}}$, $n \in \mathbb{N}$, converge to the original operation:

$$\lim_{n \to \infty} g \circ_{\psi_{p^n}} h = g \cdot h.$$

Rota–Baxter operators on groups

Let $(G, \cdot)$ be a group.

Definition ([Guo et al., 2021])

A *Rota–Baxter* operator on $(G, \cdot)$ is a map $B\colon G \to G$ such that

$$B(g \cdot B(g) \cdot h \cdot B(g)^{-1}) = B(g) \cdot B(h)$$

for all $g, h \in G$.

Proposition ([Guo et al., 2021],[Bardakov and Gubarev, 2022])

Let B be a Rota–Baxter operator on $(G, \cdot)$, and write

$$g \circ h = g \cdot B(g) \cdot h \cdot B(g)^{-1}$$

for all $g, h \in G$. Then $(G, \cdot, \circ)$ is a skew brace.

Why this is not surprising

Theorem ([Guarnieri and Vendramin, 2017])

Let $(G, \cdot)$ be a group. The following data are equivalent:

- *an operation $\circ$ such that $(G, \cdot, \circ)$ is a skew brace.*
- *a gamma function: a function $\gamma\colon G \to \mathrm{Aut}(G, \cdot)$ such that*

$$\gamma(g \cdot {}^{\gamma(g)}h) = \gamma(g)\gamma(h).$$

  for all $g, h \in G$.

Explicitly, $g \circ h = g \cdot {}^{\gamma(g)}h$.

In particular, given $C\colon G \to G$ and $g \circ h = g \cdot C(g) \cdot h \cdot C(g)^{-1}$, $(G, \cdot, \circ)$ is a skew brace if and only if for all $g, h \in G$,

$$C(g \cdot C(g) \cdot h \cdot C(g)^{-1}) \equiv C(g) \cdot C(h) \pmod{Z(G)}.$$

A natural question

Let $(G, \cdot, \circ)$ be a skew brace.

Definition

We say that $(G, \cdot, \circ)$ *comes* from a Rota–Baxter operator if there exists a Rota–Baxter operator B on $(G, \cdot)$ such that

$$g \circ h = g \cdot B(g) \cdot h \cdot B(g)^{-1}$$

for all $g, h \in G$.

Question

Do all the skew braces such that the associated gamma function has values in the inner automorphisms come from a Rota–Baxter operator?

The answer

Let $(G, \cdot, \circ)$ be such a skew brace. Then

$$g \circ h = g \cdot C(g) \cdot h \cdot C(g)^{-1},$$

where $C \colon G \to G$ satisfies

$$C(g) \cdot C(h) = \kappa(g, h) \cdot C(g \circ h)$$

for some $\kappa \colon G \times G \to Z(G)$.

Theorem ([Caranti and LS, 2022b])

- *κ is a 2-cocycle for the trivial $(G, \circ)$-module $Z(G)$, whose cohomology class in $\mathbf{H}^2((G, \circ), Z(G))$ does not depend on the choice of C.*
- *$(G, \cdot, \circ)$ comes from a Rota–Baxter operator if and only if the cohomology class of κ is trivial.*

An example

Let p be an odd prime, and let $G = \{(a, b, c) \mid a, b, c \in \mathbb{Z}/p\mathbb{Z}\}$ be the Heisenberg group of order p^3, a group of nilpotence class two.

For all $\alpha \in \{0, \dots, p-1\}$, consider

$$g \circ_\alpha h = g \cdot g^\alpha \cdot h \cdot g^{-\alpha} = g \cdot h \cdot [g, h]^\alpha.$$

Then $(G, \cdot, \circ_\alpha)$ is a skew brace.

Proposition ([Caranti and LS, 2022b])

- *If $\alpha \neq (p-2)^{-1}$, then $(G, \cdot, \circ_\alpha)$ comes from a Rota–Baxter operator.*
- *If $\alpha = (p-2)^{-1}$, then $(G, \cdot, \circ_\alpha)$ does not come from a Rota–Baxter operator.*

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