Skew braces that do not come from Rota–Baxter operators

Lorenzo Stefanello

Joint work with Andrea Caranti

Ischia Group Theory 2022, 24 June 2022

Skew braces

Definition ([Guarnieri and Vendramin, 2017])

A *skew brace* is a triple $(G, \cdot, \circ)$, where $(G, \cdot)$ and $(G, \circ)$ are groups such that

$$g \circ (h \cdot k) = (g \circ h) \cdot g^{-1} \cdot (g \circ k).$$

Here g^{-1} denotes the inverse of g in $(G, \cdot)$.

Example

- For all groups $(G, \cdot)$, $(G, \cdot, \cdot)$ is a skew brace.
- For all $a, b \in \mathbb{Z}$, define $a \circ b = a + (-1)^a b$. Then $(\mathbb{Z}, +, \circ)$ is a skew brace.

Motivations

Skew braces...

- generalise radical rings;
- yield solutions of the set-theoretic Yang–Baxter equation;
- provide regular subgroups of holomorphs of groups;
- are connected with Hopf–Galois structures.

Rota–Baxter operators on groups

Let $(G, \cdot)$ be a group.

Definition ([Guo et al., 2021])

A *Rota–Baxter* operator on $(G, \cdot)$ is a map $B\colon G \to G$ such that

$$B(g \cdot B(g) \cdot h \cdot B(g)^{-1}) = B(g) \cdot B(h).$$

Proposition ([Bardakov and Gubarev, 2022])

Let B be a Rota–Baxter operator on $(G, \cdot)$, and write

$$g \circ h = g \cdot B(g) \cdot h \cdot B(g)^{-1}.$$

Then $(G, \cdot, \circ)$ is a skew brace.

Gamma functions

Let $(G, \cdot, \circ)$ be a skew brace. For all $g \in G$, we define

$$\gamma(g) \colon G \to G, \quad h \to g^{-1} \cdot (g \circ h).$$

The function γ, called *gamma function*, is a group homomorphism

$$\gamma \colon (G, \circ) \to \mathrm{Aut}(G, \cdot).$$

Example

Suppose that

$$g \circ h = g \cdot B(g) \cdot h \cdot B(g)^{-1},$$

where B is a Rota–Baxter operator on $(G, \cdot)$. Then $\gamma(g)$ is conjugation by $B(g)$. In particular, $\gamma(G) \subseteq \mathrm{Inn}(G, \cdot)$.

A natural question

Definition

A skew brace $(G, \cdot, \circ)$ *comes* from a Rota–Baxter operator if there exists a Rota–Baxter operator B on $(G, \cdot)$ such that

$$g \circ h = g \cdot B(g) \cdot h \cdot B(g)^{-1}.$$

Question

Do all the skew braces $(G, \cdot, \circ)$ *with* $\gamma(G) \subseteq \operatorname{Inn}(G, \cdot)$ *come from Rota–Baxter operators?*

The approach

Let $(G, \cdot, \circ)$ be a skew brace with $\gamma(G) \subseteq \mathrm{Inn}(G, \cdot)$. Since $g \circ h = g \cdot {}^{\gamma(g)}h$, there exists $C\colon G \to G$ such that

$$g \circ h = g \cdot C(g) \cdot h \cdot C(g)^{-1}.$$

As $\gamma\colon (G, \circ) \to \mathrm{Aut}(G, \cdot)$ is a group homomorphism, we find that

$$C(g \circ h) \equiv C(g) \cdot C(h) \pmod{Z(G, \cdot)}.$$

In particular, there exists $\kappa\colon G \times G \to Z(G, \cdot)$ such that

$$\kappa(g, h) \cdot C(g \circ h) = C(g) \cdot C(h).$$

The main theorem

Recall:

$$g \circ h = g \cdot C(g) \cdot h \cdot C(g)^{-1},$$

and there exists $\kappa \colon G \times G \to Z(G, \cdot)$ such that

$$\kappa(g, h) \cdot C(g \circ h) = C(g) \cdot C(h).$$

Theorem ([Caranti and LS, 2022])

- *κ is a 2-cocycle for the trivial $(G, \circ)$-module $Z(G, \cdot)$, whose cohomology class in $\mathbf{H}^2((G, \circ), Z(G, \cdot))$ does not depend on the choice of C.*
- *$(G, \cdot, \circ)$ comes from a Rota–Baxter operator if and only if the cohomology class of κ is trivial.*

An example

Let p be an odd prime, and let

$$(G, \cdot) = \langle u, v, k \mid u^p, v^p, k^p, [u, v] = k, [u, k], [v, k] \rangle.$$

For all $\alpha \in \{0, \dots, p-1\}$, consider

$$g \circ_\alpha h = g \cdot g^\alpha \cdot h \cdot g^{-\alpha}.$$

Then $(G, \cdot, \circ_\alpha)$ is a skew brace.

Proposition ([Caranti and LS, 2022])

- *If $\alpha \neq (p-2)^{-1}$, then $(G, \cdot, \circ_\alpha)$ comes from a Rota–Baxter operator, which can be computed explicitly.*
- *If $\alpha = (p-2)^{-1}$, then $(G, \cdot, \circ_\alpha)$ does not come from a Rota–Baxter operator.*

Bibliography

Bardakov, V. G. and Gubarev, V. (2022).
Rota-Baxter groups, skew left braces, and the Yang-Baxter equation.
J. Algebra, 596:328–351.

Caranti, A. and LS (2022).
Skew braces from Rota–Baxter operators: a cohomological characterisation and some examples.
Annali di Matematica Pura ed Applicata (1923 -).

Guarnieri, L. and Vendramin, L. (2017).
Skew braces and the Yang-Baxter equation.
Math. Comp., 86(307):2519–2534.

Guo, L., Lang, H., and Sheng, Y. (2021).
Integration and geometrization of Rota-Baxter Lie algebras.
Adv. Math., 387:Paper No. 107834, 34.