Prova scritta del 2. 4 . 07 - Soluzioni

1.

C.E. $y \geq 1$, $x \in R$

Soluzioni costanti : $y = 1$

Altre soluzioni : separando le variabili e integrando si ottiene

$$\int_{y_0}^{y} \frac{ds}{s\sqrt{s-1}} = \int_{x_0}^{x} t\,dt$$

Nel primo integrale si pone $\sqrt{s-1} = u$, $s = u^2 + 1$, $ds = 2u\,du$:

$$\int_{\sqrt{y_0 - 1}}^{\sqrt{y-1}} \frac{du}{u^2+1} = \int_{x_0}^{x} t\,dt$$

$$2 \ \mathrm{arctg}\sqrt{y-1} - 2\,\mathrm{arctg}\sqrt{y_0 - 1} = \frac{x^2 - x_0^2}{2}$$

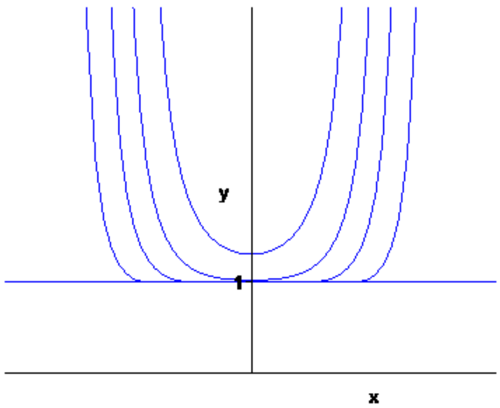
$$\mathrm{arctg}\sqrt{y-1} = \frac{x^2 - c}{4} \qquad \text{deve essere } -\frac{\pi}{2} < \frac{x^2-c}{4} < \frac{\pi}{2}$$

$$\sqrt{y-1} = \mathrm{tg}\,\frac{x^2 - c}{4} \qquad \text{deve essere } 0 < \frac{x^2-c}{4} < \frac{\pi}{2}$$

$$y = 1 + \mathrm{tg}^2\,\frac{x^2 - c}{4} \qquad \text{deve essere } c < x^2 < c + 2\pi \text{ , cioè}$$

se $-2\pi \leq c \leq 0$ $\quad |x| < \sqrt{c + 2\pi}$

se $c > 0$ $\quad \sqrt{c} < |x| < \sqrt{c + 2\pi}$

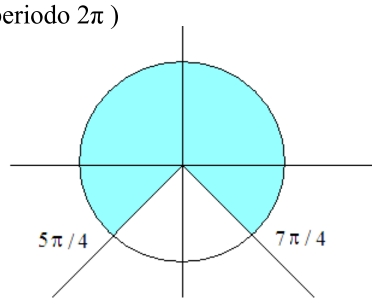


2.

C.E. $x \in [\,0\,, 2\pi\,]$ (la funzione è periodica di periodo 2π)

SGN $\mathrm{sen}\,x + |\,\cos x \geq 0$

$$Y + |X| \geq 0 \ , \ X^2 + Y^2 = 1$$

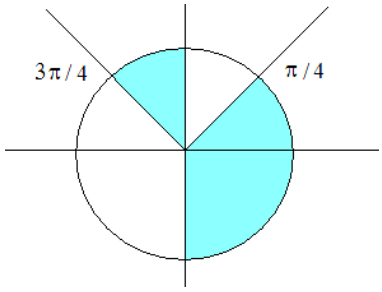


LIM nessun limite da calcolare

$f(0) = f(\pi/2) = f(\pi) = f(2\pi) = 1$, $f(3\pi/2) = -1$

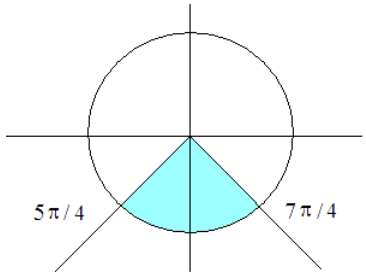
DRV $f'(x) = \cos x - \operatorname{sen} x \ \operatorname{sgn} \cos x \geq 0 \quad (x \neq \pi/2\ , x \neq 3\pi/2)$

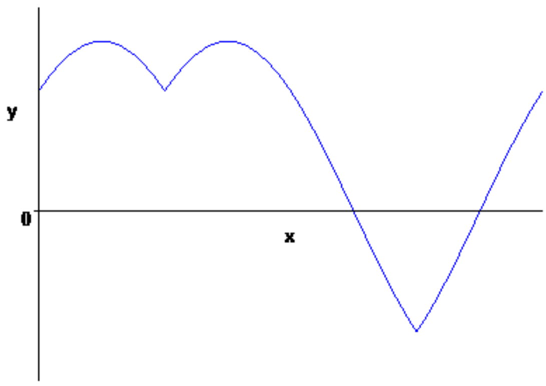
$X - Y \operatorname{sgn}(X) \geq 0 \quad , \quad X^2 + Y^2 = 1$



$f''(x) = -\operatorname{sen} x - |\cos x| \geq 0$

$-Y - |X| \geq 0 \quad , \quad X^2 + Y^2 = 1$





3.

Integrando per parti , si ottiene

$$- \cos x \log \operatorname{sen} x + \int \frac{\cos^2 x}{\operatorname{sen} x} dx$$

Per l'integrale che rimane da calcolare (funzione dispari in senx) si ottiene successivamente

$$\int \operatorname{sen} x \frac{\cos^2 x}{1 - \cos^2 x} dx \underset{\cos x = t}{=} \int \frac{t^2}{t^2 - 1} dt = \int \left(1 + \frac{1}{2} \left(\frac{1}{t-1} - \frac{1}{t+1} \right) \right) dt =$$

$$= t + \frac{1}{2} \log \left| \frac{t-1}{t+1} \right| + c = \cos x + \log \sqrt{\frac{1 - \cos x}{1 + \cos x}} + c .$$

In definitiva, dunque

$$\int \operatorname{sen} x \log \operatorname{sen} x \ dx = - \cos x \log \operatorname{sen} x + \cos x + \log \sqrt{\frac{1 - \cos x}{1 + \cos x}} + c$$

4.

(i) $a_n \approx \dfrac{n}{n+1} - 1 = -\dfrac{1}{n+1} \approx -\dfrac{1}{n}$ serie divergente

(ii) $a_n > \dfrac{1}{n^{2\alpha}}$; si sceglie $\alpha < ½$: serie divergente

5.

$$f(x) = \exp\left(\frac{1}{x^2}\log\left(\frac{tg^2x}{x\,senx}\right)\right)$$

per $x \to 0$ $$\frac{1}{x^2}\log\frac{tg^2x}{x\,senx} \approx \frac{1}{x^2}\left(\frac{tg^2x}{x\,senx} - 1\right) \approx \frac{tg^2x - x\,senx}{x^3\,senx} = \frac{5x^4/6}{x^4} \to 5/6$$

Infatti :

$tg\,x = x + x^3/3 + o(x^4)$
$tg^2x = x^2 + 2x^4/3 + o(x^4)$
$x\,senx = x^2 - x^4/6 + o(x^4)$

In conclusione , $f(x) \to e^{5/6}$