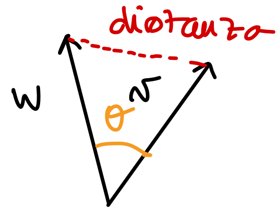
ISTITUZIONI DI MATEMATICA I, 08/05/2024

Alcuni esercizi del foglio 11

1A. $v = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ $\quad w = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

$$\Rightarrow \|v\| = \sqrt{9+16} = 5, \quad \|w\| = \sqrt{1+25} = \sqrt{26}$$



distanza: $\|v-w\| = \sqrt{(3-(-1))^2 + (4-5)^2} = \sqrt{17}$

$$v \cdot w = -3 + 20 = 17$$

$$v \cdot w = \|v\| \, \|w\| \cos\theta \quad \Rightarrow \quad \cos\theta = \frac{v \cdot w}{\|v\| \, \|w\|} = \frac{17}{5\sqrt{26}}.$$

1C. in $\mathbb{R}^3$

$v = \begin{pmatrix} 8 \\ 7 \\ -1 \end{pmatrix}$ $\quad w = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$

$$\|v\| = \sqrt{64+49+1} = \sqrt{114}$$

$$\|w\| = \sqrt{25+4+9} = \sqrt{38}$$

$$v \cdot w = 40 - 14 - 3 = 23$$

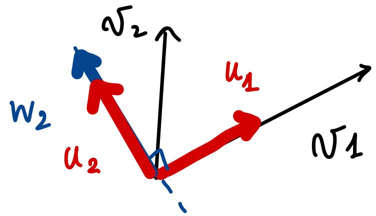
$$\cos\theta = \frac{v \cdot w}{\|v\| \, \|w\|} = \frac{23}{\sqrt{114}\sqrt{38}}.$$

$$\|v - w\| = \sqrt{9 + 81 + 16} = \sqrt{106}$$

2_08105

2 A. $v_1 = \begin{pmatrix} 5 \\ -12 \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} \sqrt{\pi} \\ -1789 \end{pmatrix}$

v_i vettori di partenza
w_i // $\perp$ tra loro
u_i // ortonormali



$$\|v_1\| = \sqrt{25+144} = \sqrt{169} = 13$$

$$\Rightarrow u_1 = \frac{v_1}{\|v_1\|} = \begin{pmatrix} 5/13 \\ -12/13 \end{pmatrix}$$ è un versore, parallelo al precedente (normalizzazione).

u_2: 1° modo: $w_2 = v_2 - (v_1 \cdot u_1)\, u_1$, $u_2 = \dfrac{w_2}{\|w_2\|}$. $(\ldots)$

2° modo: $u_2 \perp u_1 \Rightarrow u_2 = \pm \begin{pmatrix} 12/13 \\ 5/13 \end{pmatrix}$

il segno è dato da : $w_2 \cdot v_2 > 0$

$\Rightarrow \pm \frac{1}{13} (\underbrace{12\pi - 5 \cdot 1789}_{<0}) > 0 \Rightarrow$ scelgo il meno.

$$u_2 = -\begin{pmatrix} 12/13 \\ 5/13 \end{pmatrix}$$

3_08/05

2E. $v_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$

$$\|v_1\| = \sqrt{4+1+9} = \sqrt{14} \Rightarrow u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

$$w_2 = v_2 - \underbrace{(v_2 \cdot u_1)}_{\frac{10}{\sqrt{14}}} u_1 = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} - \frac{5}{7}\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 11 \\ -19 \\ -1 \end{pmatrix} \Rightarrow u_2 = \frac{w_2}{\|w_2\|} = \frac{1}{\sqrt{483}}\begin{pmatrix} 11 \\ -19 \\ -1 \end{pmatrix}$$

$w_2 \perp u_1$: $11 \cdot 2 - 19 \cdot 1 - 3 = 0$

2F. $v_1 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ $v_2 = \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix}$ $v_3 = \begin{pmatrix} \pi \\ 0 \\ \sqrt{3} \end{pmatrix}$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{14}}\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$v_2 \cdot u_1 = \frac{1}{\sqrt{14}}(2 - 5 - 6) = -\frac{9}{\sqrt{14}}$$

$$w_2 = v_2 - (v_2 \cdot u_1) u_1 = \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix} + \frac{9}{14}\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 32 \\ 61 \\ -1 \end{pmatrix}$$

check: $u_1 \perp w_2$:
$2 \cdot 32 - 61 - 3 = 0$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{1}{\sqrt{4746}}\begin{pmatrix} 32 \\ 61 \\ -1 \end{pmatrix}$$

4-08/05

u_3: 1° modo: formula $\quad w_3 = v_3 - (v_3 \cdot u_1)u_1 - (v_3 \cdot u_2)u_2, \quad u_3 = \dfrac{w_3}{\|w_3\|}$

2° modo: $\begin{cases} u_3 \perp u_1 \\ u_3 \perp u_2 \end{cases} \Rightarrow u_3 \,/\!/\, (u_1 \times u_2)$

$$u_1 \times u_2 = \overbrace{\frac{1}{\sqrt{14}}\,\frac{1}{\sqrt{4746}}}^{\frac{1}{14\cdot\sqrt{339}}} \boxed{\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 32 \\ 61 \\ -1 \end{pmatrix}} \underset{\text{regola dei determinanti}}{=} \frac{1}{14\cdot\sqrt{339}} \begin{pmatrix} -182 \\ 98 \\ 154 \end{pmatrix}$$

Poiché u_1 e u_2 sono due versori ortogonali, $u_1 \times u_2$ è un versore

$\Rightarrow u_3 = \pm (u_1 \times u_2)$. Il segno è dato da $u_3 \cdot v_3 > 0$

infatti $\quad v_3 \cdot u_3 = v_3 \cdot \dfrac{1}{\|w_3\|}\left(v_3 - (v_3 \cdot u_1)u_1 - (v_3 \cdot u_2)u_2\right)$

$$= \frac{1}{\|w_3\|}\left(\|v_3\|^2 - \|v_3 \cdot u_1\|^2 - \|v_3 \cdot u_2\|^2\right) > 0$$

$$v_3 \cdot (u_1 \times u_2) = \frac{1}{14}\,\frac{1}{\sqrt{339}}\left(-182\pi + 154\sqrt{3}\right) < 0$$

$$\Rightarrow u_3 = -(u_1 \times u_2) = \boxed{-\frac{1}{14}\,\frac{1}{\sqrt{339}}\begin{pmatrix} -182 \\ 98 \\ 154 \end{pmatrix}}$$

5_08/05

3. $W = \text{span}(5e_1 - 2e_2 + 4e_3 + e_4,\ 3e_1 + 7e_2 - e_3 - 2e_4)$

$$= \text{span}\left(\overset{w_1}{\begin{pmatrix} 5 \\ -2 \\ 4 \\ 1 \end{pmatrix}}, \overset{w_2}{\begin{pmatrix} 3 \\ 7 \\ -1 \\ -2 \end{pmatrix}} \right)$$

$\dim W = 2$ in $\mathbb{R}^4 \Rightarrow \dim W^\perp = 2$ $\qquad v \in W^\perp \Leftrightarrow \begin{cases} v \cdot w_1 = 0 \\ v \cdot w_2 = 0 \end{cases}$

$$v = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in W^\perp \Leftrightarrow \begin{cases} 5x - 2y + 4z + t = 0 \\ 3x + 7y - z - 2t = 0 \end{cases}$$

$$\begin{cases} t = -5x + 2y - 4z \\ 3x + 7y - z + 10x - 4y + 8z = 0 \end{cases} \qquad \begin{cases} t = -\frac{41}{3}x - \frac{26}{3}z \\ y = -\frac{13}{3}x - \frac{7}{3}z \end{cases}$$

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x \\ -13/3\,x - 7/3\,z \\ z \\ -\frac{41}{3}x - \frac{26}{3}z \end{pmatrix} = x \begin{pmatrix} 1 \\ -13/3 \\ 0 \\ -41/3 \end{pmatrix} + z \begin{pmatrix} 0 \\ -7/3 \\ 1 \\ -26/3 \end{pmatrix}$$

Base di $W^\perp$ $\left(\begin{pmatrix} 3 \\ -13 \\ 0 \\ -41 \end{pmatrix}, \begin{pmatrix} 0 \\ -7 \\ 3 \\ -26 \end{pmatrix} \right)$

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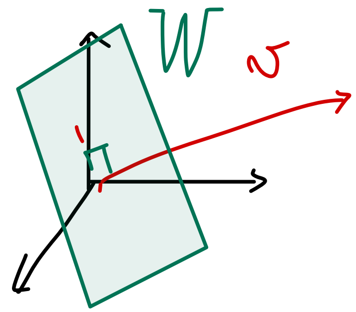
5c. P matrice di proiezione ortogonale di $\mathbb{R}^n$ su W.



$$W = \{ x \in \mathbb{R}^3 : x_1 - 2x_2 + 5x_3 = 0 \} = \Big(\overbrace{\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}}^{v} \Big)^{\perp}$$

$$= \operatorname{span} \left(\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$\underbrace{P_W}_{\text{funzione}} (x) = \underbrace{P}_{\text{matrice}} x \qquad P = ?$$

$$P_W(x) = x - \left(x \cdot \frac{v}{\|v\|} \right) \frac{v}{\|v\|} = x - \frac{(x \cdot v)}{\|v\|^2} v \overset{\|v\|^2 = 30}{=} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \frac{1}{30} (x_1 - 2x_2 + 5x_3) \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$$

$$= \frac{1}{30} \begin{pmatrix} 30x_1 - x_1 + 2x_2 - 5x_3 \\ 30x_2 + 2x_1 - 4x_2 + 10x_3 \\ 30x_3 - 5x_1 + 10x_2 - 25x_3 \end{pmatrix} = \underbrace{\frac{1}{30} \begin{pmatrix} 29 & 2 & -5 \\ 2 & 26 & 10 \\ -5 & 10 & 5 \end{pmatrix}}_{P} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

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$$P^2 = \begin{pmatrix} \frac{870}{900} & \frac{60}{900} & \frac{-150}{30} \\ \frac{60}{900} & \frac{780}{900} & \frac{300}{900} \\ \frac{-150}{900} & \frac{300}{900} & \frac{150}{900} \end{pmatrix} = P \,, \quad {}^tP = P$$

5h. $W = \text{span} \underbrace{\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}}_{w}$ in $\mathbb{R}^3$

$$\|w\| = \sqrt{4+1+25} = \sqrt{30} \qquad x \cdot w = 2x_1 - x_2 + 5x_3$$

$$P_W(x) = \left(x \cdot \frac{w}{\|w\|}\right) \frac{w}{\|w\|} = (x \cdot w) \frac{w}{\|w\|^2} = \frac{1}{30}(2x_1 - x_2 + 5x_3) \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$$

$$= \frac{1}{30} \begin{pmatrix} 4x_1 - 2x_2 + 10x_3 \\ -2x_1 + x_2 - 5x_3 \\ 10x_1 - 5x_2 + 25x_3 \end{pmatrix} = \underbrace{\frac{1}{30} \begin{pmatrix} 4 & -2 & 10 \\ -2 & 1 & -5 \\ 10 & -5 & 25 \end{pmatrix}}_{P} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$${}^tP = P, \qquad P^2 = P$$

8 - 08/05

6A.

$$\begin{pmatrix}4\\1\\-2\end{pmatrix}\times\begin{pmatrix}-1\\3\\5\end{pmatrix}$$ 1° el. $= +(5-(-6)) = 11$

$$\begin{pmatrix}4\\1\\-2\end{pmatrix}\times\begin{pmatrix}-1\\3\\5\end{pmatrix}$$ 2° el. $= -(20-2) = -18$

$$\begin{pmatrix}4\\1\\-2\end{pmatrix}\times\begin{pmatrix}-1\\3\\5\end{pmatrix}$$ 3° el. $= +(12+1) = 13$

$$\Rightarrow \begin{pmatrix}4\\1\\-2\end{pmatrix}\times\begin{pmatrix}-1\\3\\5\end{pmatrix} = \boxed{\begin{pmatrix}11\\-18\\13\end{pmatrix}}$$

7E. $W \subset \mathbb{R}^3$ W in forma parametrica è $\begin{cases} x = 4t \\ y = -3t \\ z = 6t \end{cases}$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = t\begin{pmatrix}4\\-3\\6\end{pmatrix} \Rightarrow W = \text{span}\begin{pmatrix}4\\-3\\6\end{pmatrix}$$

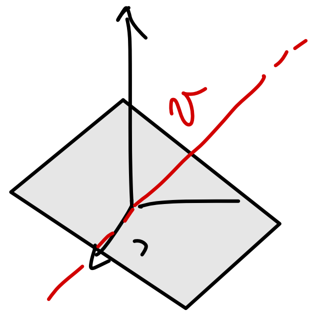
in forma cartesiana è $t = x/4$ $y = -\frac{3}{4}x$ $z = \frac{3}{2}x$ $\boxed{\begin{cases} 4y + 3x = 0 \\ 2z - 3x = 0 \end{cases}}$

9 _08/05

7H. $W := \{ 5x - 12y + 10z = 0 \} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -12 \\ 10 \end{pmatrix} = 0 \right\} = \begin{pmatrix} 5 \\ -12 \\ 10 \end{pmatrix}^{\perp}$ piano

$W \subset \mathbb{R}^3$ $\quad$ $\dim W = 2$

in forma parametrica è $\boxed{\operatorname{span}\left(\begin{pmatrix} 12 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 10 \\ 12 \end{pmatrix} \right)}$.



8B. Piano $W = \operatorname{span}\left(\begin{pmatrix} -2 \\ 4 \\ -3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} \right)$

$v \perp W$

$\Rightarrow v \in \operatorname{span}\left(\underbrace{\begin{pmatrix} -2 \\ 4 \\ -3 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}}_{\text{1 vettore } \perp \text{ ai due generatori di } W} \right) = \boxed{\operatorname{span}\begin{pmatrix} 23 \\ 16 \\ 6 \end{pmatrix}}$

$\Rightarrow$ genera $W^{\perp}$

10 – 08/05

9 C. Eq. cartesiana del piano $\perp$ alla retta $\begin{cases} -4x+7y-5z=0 \\ 2x-5y-4z=0 \end{cases}$

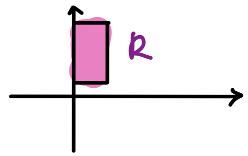
Retta : $\left(\begin{pmatrix} -4 \\ 7 \\ -5 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ -4 \end{pmatrix} \right)^{\perp}$ eq. cartesiana

Piano : $\text{Retta}^{\perp} = \text{span}\left(\begin{pmatrix} -4 \\ 7 \\ -5 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ -4 \end{pmatrix} \right)$ forma parametrica

$= (v)^{\perp}$ per quale v?

Ad esempio $v = \begin{pmatrix} -4 \\ 7 \\ -5 \end{pmatrix} \times \begin{pmatrix} 2 \\ -5 \\ -4 \end{pmatrix} = \begin{pmatrix} -53 \\ -26 \\ 6 \end{pmatrix} \Longrightarrow$ Piano : $\begin{pmatrix} -53 \\ -26 \\ 6 \end{pmatrix}^{\perp}$ eq. cartesiana

10 A. Integrale doppio su un rettangolo $R = \{ 0 \le x \le 5,\ 2 \le y \le 10 \}$



$$\iint_R \cos(x-y)\,dx\,dy = \int_0^5 \left(\int_2^{10} \cos(x-y)\,dy \right) dx = \int_0^5 \Big[-\sin(x-y) \Big|_{y=2}^{y=10} dx$$

$$= \int_0^5 \left[-\sin(x-10) + \sin(x-2)\right] dx = \left[\cos(x-10) - \cos(x-2)\right|_{x=0}^{x=5}$$

$$= \cos(-5) - \cos(3) - \cos(-10) + \cos(-2)$$

Equivalentemente, avremmo potuto integrare prima in x e poi in y:

$$\iint \cdots = \int_2^{10} \left(\int_0^5 \cos(x-y)\, dx \right) dy = \int_2^{10} \left[\sin(x-y) \right|_{x=0}^{x=5} dy$$

$$= \int_2^{10} \left[\sin(5-y) - \sin(-y)\right] dy = \int_2^{10} \left[-\sin(y-5) + \sin y\right] dy = \left[\cos(y-5) - \cos y\right|_{y=2}^{y=10}$$

$$= \cos(5) - \cos(10) - \cos(-3) + \cos(2)$$

110. D = regione di piano delimitata dalle curve $y = \sqrt{x}$, $y = 1$, $x = 0$.

D è semplice nelle due direzioni:

$$D = \{0 \le x \le 1,\ \sqrt{x} \le y \le 1\} = \{0 \le y \le 1,\ 0 \le x \le y^2\}$$

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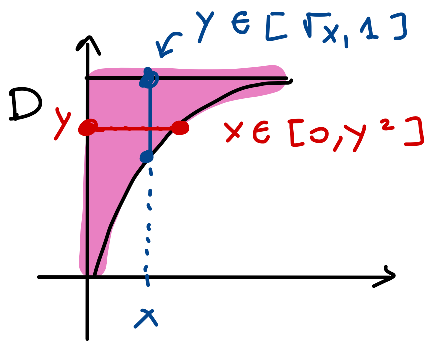


primitiva difficile in y

$$\iint_D \sin(y^3)\,dy = \begin{cases} \int_0^1 \int_{\sqrt{x}}^1 \sin(y^3)\,dy \quad dx \\ \int_0^1 \int_0^{y^2} \sin(y^3)\,dx \quad dy \end{cases}$$

primitiva facile in x

$$\int \sin(y^3)\,dx = x\sin(y^3) + C(y) \quad \text{infatti} \quad \frac{\partial}{\partial x}\left(x\sin(y^3) + C(y)\right) = \sin(y^3)$$

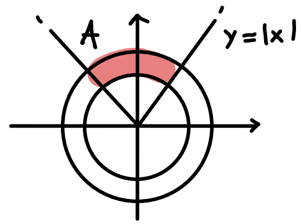
$$\Rightarrow \int_0^1 \left(\int_0^{y^2} \sin(y^3)\,dx \right) dy = \int_0^1 \left[\sin(y^3)\,x \right]_{x=0}^{x=y^2} dy = \frac{1}{3}\int_0^1 3\sin(y^3)\,y^2\,dy$$

$$= \frac{1}{3}\left[-\cos(y^3) \right]_{y=0}^{y=1} = \frac{1}{3}\left(1 - \cos(1)\right)$$

12 A.

$$A = \{ 1 \le \underbrace{x^2 + y^2} \le 4, \quad y \geq |x| \}$$

distanza da (0,0) al quadrato



A in coordinate polari è $R = \{ 1 \le \rho \le 2, \quad \pi/4 \le \theta \le 3/4\,\pi \}$

13-08/05

Cambio di variabile:

$$\iint_A x^2 y \, dx dy = \iint_R \underbrace{\rho^2 \cos^2\theta \, \rho \sin\theta}_{x^2 y} \, \underbrace{\rho \, d\rho \, d\theta}_{dx\,dy} = \int_{\pi/4}^{3\pi/4} \int_1^2 \rho^4 \cos^2\theta \sin\theta \, d\rho \, d\theta$$

$$x = \rho\cos\theta$$
$$y = \rho\sin\theta$$
$$dx\,dy = \rho\, d\rho\, d\theta$$

$$= \int_{\pi/4}^{3\pi/4} \left[\frac{\rho^5}{5} \cos^2\theta \sin\theta \right]_{\rho=1}^{\rho=2} d\theta$$

$$= \int_{\pi/4}^{3\pi/4} \left(\frac{32}{5} - \frac{1}{5} \right) \cos^2\theta \sin\theta \, d\theta = \frac{31}{5} \left[-\frac{\cos^3\theta}{3} \right]_{\theta=\pi/4}^{\theta=3\pi/4} = \frac{31}{30}\sqrt{2}.$$