Ist. Mat. I - C/A
21/3/24

$f: V \to W$ lin; $\operatorname{Ker}(f) = \{v \in V : f(v) = 0\}$
$\operatorname{Im}(f) = \{f(v) : v \in V\}$

Teo: $\underbrace{\dim(V)}_{n} = \underbrace{\dim(\operatorname{Ker}(f))}_{k} + \dim(\operatorname{Im}(f))$

Schema dim:

prendo base $v_1, \dots, v_k$ del nucleo

completo a base $v_1, \dots, v_k, v_{k+1}, \dots, v_n$ di V

dimostro che $\underbrace{f(v_{k+1}), \dots, f(v_n)}_{n-k}$ sono base di $\operatorname{Im}(f)$

- lin. indip. ....

- generano: se $v_1, \dots, v_n$ è base di V allora $f(v_1), \dots, f(v_n)$ generano $\operatorname{Im}(f)$:

$$w \in \operatorname{Im}(f) \Longrightarrow w = f(v) = f(\alpha_1 v_1 + \dots + \alpha_n v_n)$$
$$= \alpha f(v_1) + \dots + \alpha_n f(v_n)$$

Per la base di sopra:

$$\underset{\substack{\| \\ 0}}{f(v_1)}, \dots, \underset{\substack{\| \\ 0}}{f(v_k)}, \underbrace{f(v_{k+1}), \dots, f(v_n)}_{\text{generano.}}$$

$\boxtimes$

Oss: se $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ allora $A \cdot e_i^{(n)}$ è la colonna i-esima di A.

$$\underset{\substack{\parallel \\ A \\ \cap \\ \mathcal{M}_{2\times 3}}}{\begin{pmatrix} 4 & \boxed{\begin{matrix} 7 \\ \sqrt{5} \end{matrix}} & -2 \\ \pi & & 11/3 \end{pmatrix}} \cdot \underset{\substack{\parallel \\ e_2^{(3)}}}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} = \boxed{\begin{pmatrix} 7 \\ \sqrt{5} \end{pmatrix}}$$

Conseguenza: le colonne di A generano $\mathrm{Im}(A)$.

Es: $f : \mathbb{R}^3 \to \mathbb{R}^4$ $\quad f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & -1 & 7 \\ -4 & 2 & -11 \\ 5 & 1 & 5 \\ 1 & -3 & 9 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\mathrm{Ker}(f): \begin{cases} 3x - y + 7z = 0 \\ -4x + 2y - 11z = 0 \leftarrow \\ 5x + y + 5z = 0 \\ x - 3y + 9z = 0 \leftarrow \end{cases}$$

$$\begin{matrix} \mathrm{I} \\ \mathrm{III} \\ \\ \\ \end{matrix} \begin{cases} y = 3x + 7z \\ 5x + 3x + 7z + 5z = 0 \\ \dots \\ \dots \end{cases} \qquad \begin{matrix} \mathrm{II} \\ \mathrm{I} \\ \\ \\ \end{matrix} \begin{cases} z = -\frac{2}{3}x \\ y = 3x - \frac{14}{3}x = -\frac{5}{3}x \\ -4x - \frac{10}{3}x + \frac{22}{3}z = 0 \quad \underline{\underline{\text{OK}}} \\ x + 5x - 6x = 0 \quad \underline{\underline{\text{OK}}} \end{cases}$$

$$\mathrm{Ker}(f) = \mathrm{Span}\begin{pmatrix} -3 \\ 5 \\ 2 \end{pmatrix} \qquad \dim = 1$$

$$\underset{\substack{\parallel \\ 3}}{\dim(\mathbb{R}^3)} = \underset{\substack{\parallel \\ 1}}{\dim(\mathrm{Ker}(f))} + \dim(\mathrm{Im}(f))$$

So che $Im(f)$ è generata dalle col. di A:

$$\begin{pmatrix} & A & \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 5 \\ 2 \end{pmatrix} = 0$$

significa: $\quad I \text{ col} = \dfrac{5 \cdot II + 2 \cdot III}{3}$

Base di $Im(f)$ è $\left(\underset{f(e_2)}{\underset{\shortparallel}{II}}, \underset{f(e_3)}{\underset{\shortparallel}{III}}\right)$

$$\underbrace{\underbrace{\begin{pmatrix} -3 \\ 5 \\ 2 \end{pmatrix}}_{\text{base Ker}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{base di } \mathbb{R}^3}$$

$$\underbrace{\qquad\qquad\qquad}$$
$$\downarrow f$$
base di $Im(f)$.

Calcolo diff. in più variabili

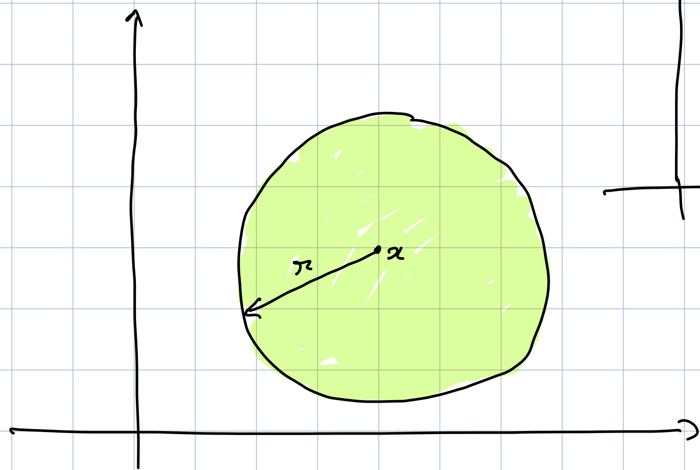
In una:

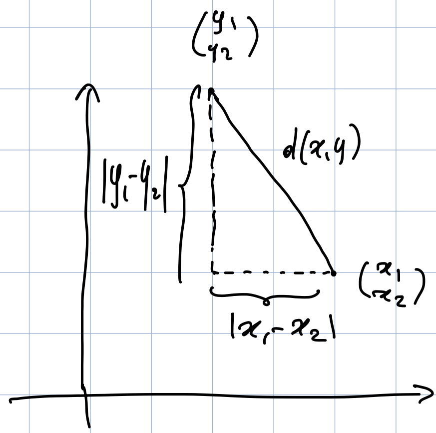
- $f : [a,b] \longrightarrow \mathbb{R}$ continua ha max/min ass.
- $f : (a,b) \longrightarrow \mathbb{R} \qquad f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(a)}{h}$

$$f(x+t) = f(x) + t \cdot f'(a) + o(t)$$

$\mathbb{R}^n$; $d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$

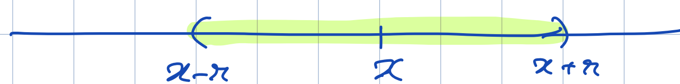
$B(x,r) = \{ y \in \mathbb{R}^n : d(x,y) < r \}$

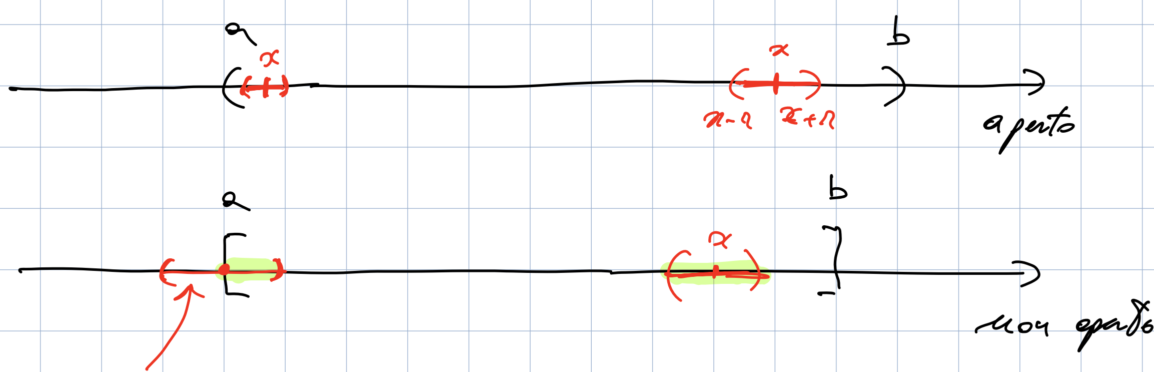


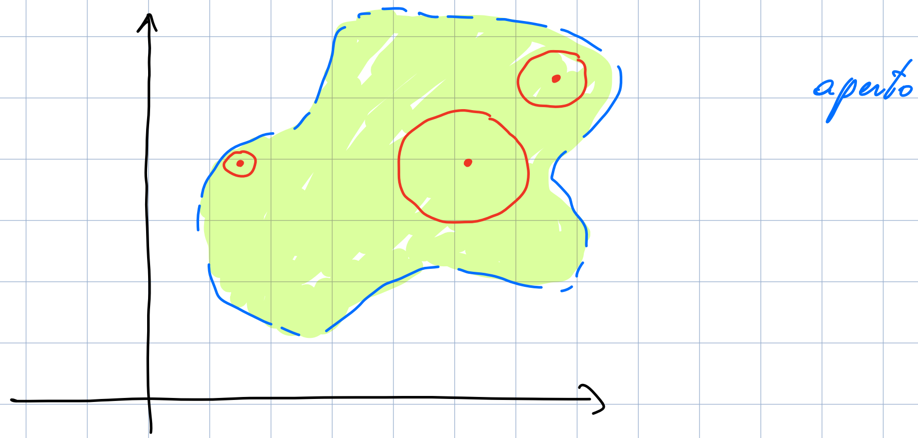


Def: $A \subset \mathbb{R}^n$ è *aperto* se $\forall x \in A \ \exists r > 0$ t.c. $B(x,r) \subset A$.

$n = 1$ $\quad B(x,r) = (x-r, x+r)$

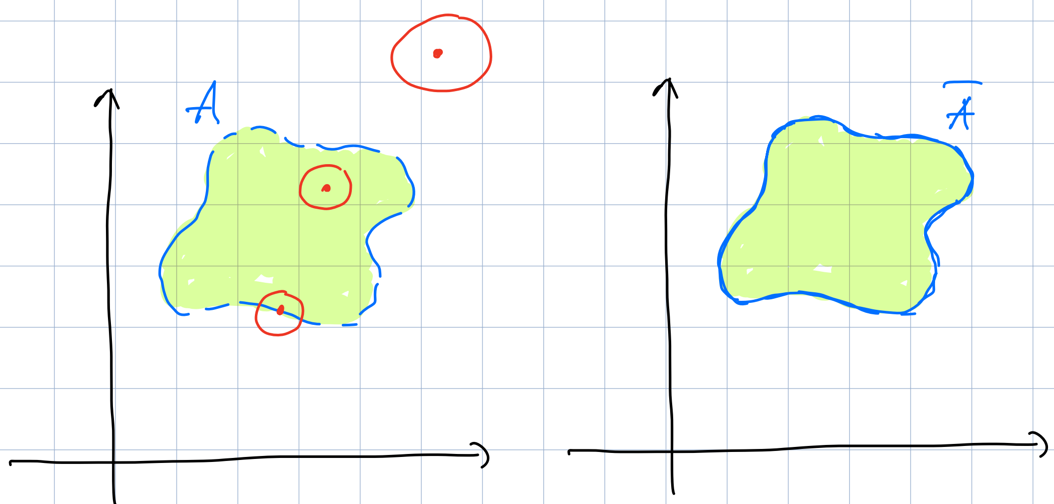






Def: dato $A \subset \mathbb{R}^n$ chiamo chiusura di A

$$\overline{A} = \{x \in \mathbb{R}^n : \quad \forall r > 0, \quad B(x,r) \cap A \neq \emptyset\}$$

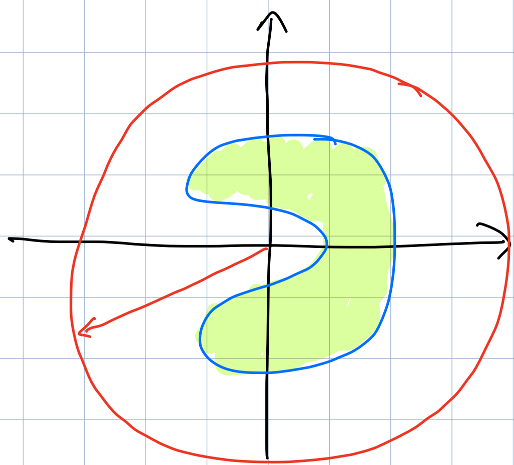


Dico che A è chiuso se $\overline{A} = A$.

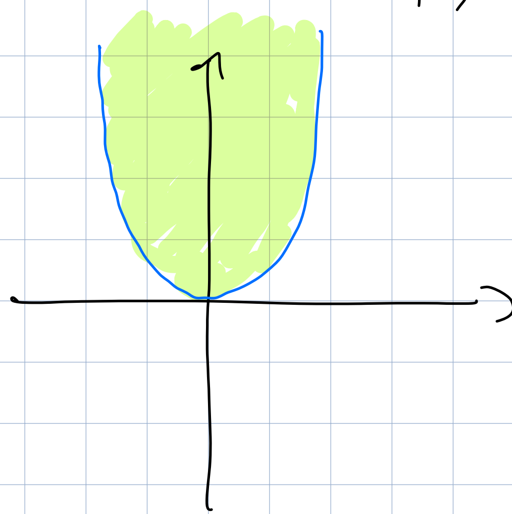
Esercizio : $\overline{(a,b)} = [a,b]$

quindi un intervallo è chiuso se e solo se è $[a,b]$.

$A \subset \mathbb{R}^n$ è limitato se $\exists r$ t.c. $A \subset B(0,r)$



limitato



non lim.

$f : A \to \mathbb{R}$ è continua in x se

$\exists \varepsilon > 0 \quad \exists \delta > 0 \quad$ t.c. $|f(x) - f(y)| < \varepsilon$
$\forall y \in A$ t.c. $d(x,y) < \delta$.

<u>Fatto</u>: A chiuso e limitato, $f: A \to \mathbb{R}$ continua allora f ha max/min ass.

——— o ———

Def: Se $A \subset \mathbb{R}^n$ è aperto e $f: A \to \mathbb{R}$, $x \in A$ chiamo derivata parziale j-esima di f, indicata
$\frac{\partial f}{\partial x_j}$ = derivata rispetto a x_j considerando le altre variabili come parametri fissi.

Es: $f:\mathbb{R}^2 \to \mathbb{R}$; $f\begin{pmatrix} x \\ y \end{pmatrix} = x^3 \cdot \cos(y^2)$

$$\frac{\partial f}{\partial x} = 3x^2 \cdot \cos(y^2) \qquad \frac{\partial f}{\partial y} = x^3 \cdot (-2y \cdot \sin(y^2))$$

Es: $f:\mathbb{R}^3 \to \mathbb{R}$, $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{x^2 z^3 y^4} \cdot \sin(5x + y^2 - 11z^3)$

$$\frac{\partial f}{\partial x} \qquad \frac{\partial f}{\partial y} \qquad \frac{\partial f}{\partial z} = 3x^2 z^2 y^4 \cdot e^{x^2 z^3 y^4} \cdot \sin(5x + y^2 - 11z^3) + e^{x^2 z^3 y^4} \cdot \cos(5x + y^2 - 11z^3) \cdot (-33z^2)$$

Taylor I in più variabili:

$n=2$: $$f\left(\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}\right) = f\begin{pmatrix} x \\ y \end{pmatrix} + \frac{\partial f}{\partial x}\begin{pmatrix} x \\ y \end{pmatrix} \cdot a + \frac{\partial f}{\partial y}\begin{pmatrix} x \\ y \end{pmatrix} \cdot b + o\left(\sqrt{a^2 + b^2}\right)$$

n qualsiasi: $$f(x+v) = f(x) + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x) \cdot v_j + o\left(\sqrt{v_1^2 + \dots + v_n^2}\right)$$

Funzioni a valori vettoriali:

$$A \xrightarrow{f} \mathbb{R}^m, \quad A \subset \mathbb{R}^n \qquad f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$$

Es: $f:\mathbb{R}^2 \to \mathbb{R}^3$ $\quad f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 y^3 \\ \cos(2xy^7) \\ e^{5x} \cdot y \end{pmatrix}$ $\quad \otimes$

Matrice jacobiana $Jf(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \in M_{m\times n}(\mathbb{R})$

$m = 1$ $\quad Jf = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$ differenziale / gradiente

Es: ⊗ $\quad Jf = \begin{pmatrix} \boxed{} & \boxed{} \\ \boxed{} & \boxed{-14y^6 \sin(2xy^7)} \\ \boxed{5e^{5x} \cdot y} & \boxed{} \end{pmatrix}$

$n = 1$ $\quad (a,b) \xrightarrow{f} (c,d) \xrightarrow{g} \mathbb{R}$

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$$

$$A \xrightarrow{f} B \xrightarrow{g} \mathbb{R}^k$$

$A \subset \mathbb{R}^n$, $\quad B \subset \mathbb{R}^m$

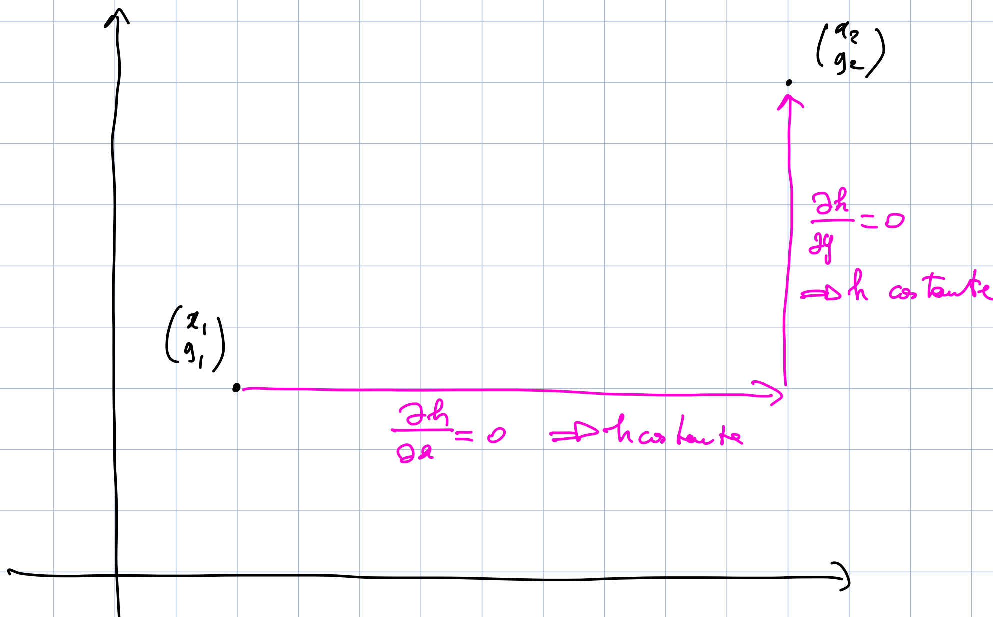
$$\underbrace{J(g\circ f)(x)}_{k \times n} = \underbrace{\underbrace{Jg(f(x))}_{k \times m} \cdot \underbrace{Jf(x)}_{m \times n}}_{k \times n}$$

$n=1$: $f,g:(a,b)\to\mathbb{R}$, $f'(x)=g'(x)\ \forall x \implies g = f + \text{cost.}$

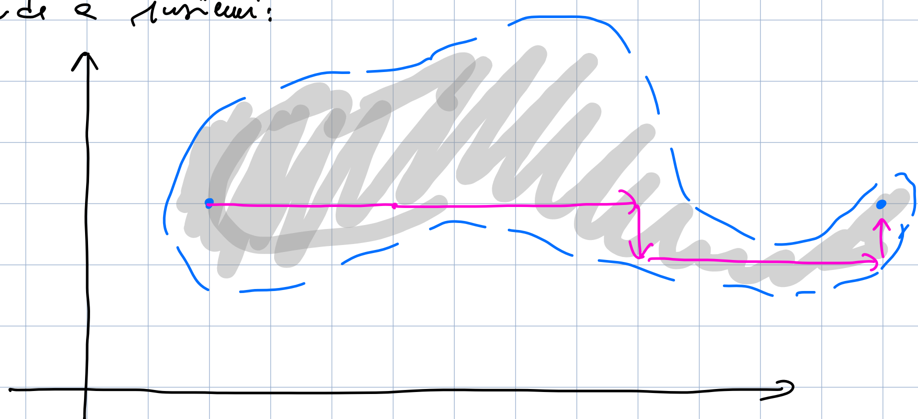
Oss: se $f,g:\mathbb{R}^n \longrightarrow \mathbb{R}$ e $\dfrac{\partial f}{\partial x_j}(x) = \dfrac{\partial g}{\partial x_j}(x)\ \forall x\ \forall j$

$\implies g = f + \text{cost}$

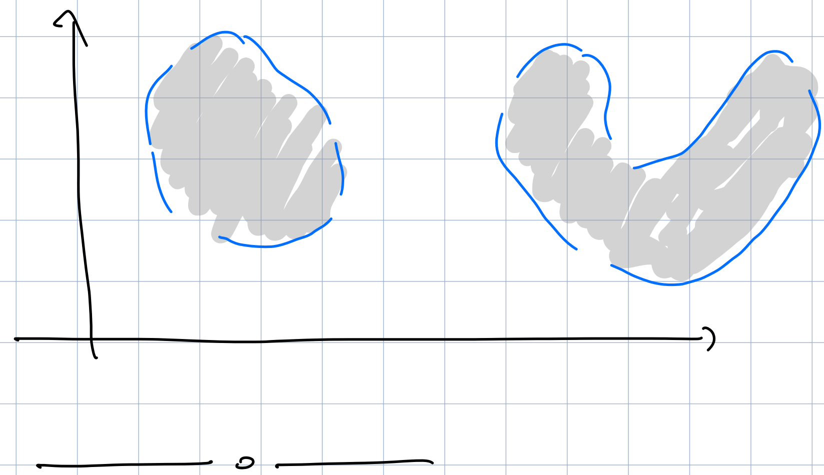
Ragione: basta vedere che se $\dfrac{\partial h}{\partial x_j}(x) = 0\ \forall x, j$
$\implies h$ costante



Si estende a funzioni:



Non è



Visto:

- V, W sp. vett. $f: V \to W$ lin. se rispetta comb. lin.
- le $f: \mathbb{R}^n \to \mathbb{R}^m$ lineari sono tutte e sole quelle $f(x) = A \cdot x$ con $A \in M_{m \times n}(\mathbb{R})$.

Def: data $f: V \to W$ lineare e basi $\mathcal{B} = (v_1, \dots, v_n)$ di V, $\mathcal{C} = (w_1, \dots, w_m)$ di W chiamo matrice associata ad f rispetto a $\mathcal{B}$ in partenza e $\mathcal{C}$ in arrivo la matrice $A \in M_{m \times n}$ $A = (a_{ij})$ t.c.

$$f(v_j) = \sum_{i=1}^{m} a_{ij} w_i \, .$$

La indico con $[f]_{\mathcal{B}}^{\mathcal{C}}$.

- prendo j-esimo elemento della base in partenza
- calcolo la sua immagine
- trovo le sue coord. rispetto alla base in arrivo
- le scrivo nella j-esima colonna.

Es: $f: \mathbb{R}^2 \to \mathbb{R}^3$ $\quad f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x - 3y \\ 2x + y \\ -6x + 7y \end{pmatrix}$

(Oss: f è l'applicaz. associata alla matrice $\begin{pmatrix} 5 & -3 \\ 2 & 1 \\ -6 & 7 \end{pmatrix}$.)

$\mathcal{B} = \left(\begin{pmatrix} 2 \\ -5 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right)$

$\mathcal{C} = \left(\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \right)$

$[f]_{\mathcal{B}}^{\mathcal{C}} \in \mathcal{M}_{3\times 2}$

I col: $\begin{pmatrix} 2 \\ -5 \end{pmatrix} \rightsquigarrow f\begin{pmatrix} 2 \\ -5 \end{pmatrix} = \begin{pmatrix} 25 \\ -1 \\ -47 \end{pmatrix}$

$\rightsquigarrow \begin{pmatrix} 25 \\ -1 \\ -47 \end{pmatrix} = a_{11} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + a_{21} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + a_{31} \cdot \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$

$\begin{cases} 2a_{21} + a_{31} = 25 \\ a_{11} + a_{21} = -1 \\ -a_{11} + 3a_{31} = -47 \end{cases} \qquad \begin{cases} a_{31} = 25 - 2a_{21} \\ a_{11} = -1 - a_{21} \\ 1 + a_{21} + 75 - 6a_{21} = -47 \end{cases}$

$\begin{cases} a_{21} = 123/5 \\ a_{11} = -128/5 \\ a_{31} = 25 - \frac{246}{5} = -\frac{121}{5} \end{cases}$

$\Rightarrow [f]_{\mathcal{B}}^{\mathcal{C}} = \begin{pmatrix} -128/5 & * \\ 123/5 & * \\ -121/5 & * \end{pmatrix}$

stesso partendo da $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

Es: $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x+y+z=0 \right\}$

$f : \mathbb{R}^2 \longrightarrow W \qquad f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - 7y \\ 5x + 8y \\ -7x - y \end{pmatrix}$

Oss: $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -7 \\ 5 & 8 \\ -7 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$

l'applicazione g associata a $\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$ va da $\mathbb{R}^2$ a $\mathbb{R}^3$

la definizione di f va bene se è vero che

$$g\begin{pmatrix} x \\ y \end{pmatrix} \in W \qquad \forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

vero: $(2x - 7y) + (5x + 8y) + (-7x - y) = 0 \quad \forall \begin{pmatrix} x \\ y \end{pmatrix}$

$\mathcal{B} = \left(\begin{pmatrix} 2 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right) \qquad \mathcal{C} = \left(\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right)$

$\begin{pmatrix} 2 \\ -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & -7 \\ 5 & 8 \\ -7 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 11 \\ 2 \\ -13 \end{pmatrix} = \boxed{11} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \boxed{2} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & -7 \\ 5 & 8 \\ -7 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -19 \\ 29 \\ -10 \end{pmatrix} = \boxed{-19} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \boxed{29} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$$[f]_{\mathcal{B}}^{\mathcal{C}} = \begin{pmatrix} 11 & -19 \\ 2 & 29 \end{pmatrix}$$

Proprieta' naturali :

- $A \in M_{m\times n}(\mathbb{R}) \rightsquigarrow f_A : \mathbb{R}^n \to \mathbb{R}^m$
$$x \mapsto A\cdot x$$

$$[f_A]^{\mathcal{E}^{(m)}}_{\mathcal{E}^{(n)}} = A$$

Es: $f: \mathbb{R}^2 \to \mathbb{R}^3 \quad f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x - 3y \\ 2x + y \\ -6x + 7y \end{pmatrix} \quad A = \begin{pmatrix} 5 & -3 \\ 2 & 1 \\ -6 & 7 \end{pmatrix}$

$[f]^{\mathcal{E}^{(3)}}_{\mathcal{E}^{(2)}} \qquad \mathcal{E}^{(2)} = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \quad \mathcal{E}^{3} = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$

$$f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -6 \end{pmatrix} = 5\cdot\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2\cdot\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 6\cdot\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 7 \end{pmatrix} = -3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1\cdot\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 7\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$[f]^{\mathcal{E}^{(3)}}_{\mathcal{E}^{(2)}} = \begin{pmatrix} 5 & -3 \\ 2 & 1 \\ -6 & 7 \end{pmatrix}$$

- $f: V \longrightarrow W$

V: dim n, $\mathcal{B} = (v_1, \dots, v_n)$

W: dim m, $\mathcal{C} = (w_1, \dots, w_m)$

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "{[\cdot]_{\mathcal{B}}}"'] \arrow[dr, bend left] & W \arrow[d, "{[\cdot]_{\mathcal{C}}}"] \\ \mathbb{R}^n \arrow[r, "{[f]^{\mathcal{C}}_{\mathcal{B}}}"'] & \mathbb{R}^m \end{tikzcd}$$

$$[f]^{\mathcal{C}}_{\mathcal{B}} \in M_{m\times n}$$

Fatto: $[f(v)]_{\mathcal{C}} = [f]^{\mathcal{C}}_{\mathcal{B}} \cdot [v]_{\mathcal{B}}$

- l'azione di f sui vettori coincide con l'azione della matrice di f sulle coordinate (se uso le stesse basi)

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[rr, bend left, "g\circ f"] & W \arrow[r, "g"] & Z \end{tikzcd}$$

$\mathcal{B}=(v_1,\dots,v_n)$ $\quad$ $\mathcal{C}=(w_1,\dots,w_m)$ $\quad$ $\mathcal{D}=(z_1,\dots,z_k)$

$[f]^{\mathcal{C}}_{\mathcal{B}} \in M_{m\times n}$ $\qquad$ $[g]^{\mathcal{D}}_{\mathcal{C}} = k\times m$

$[g\circ f]^{\mathcal{D}}_{\mathcal{B}} = k\times n$

Fatto: $[g\circ f]^{\mathcal{D}}_{\mathcal{B}} = [g]^{\mathcal{D}}_{\mathcal{C}} \cdot [f]^{\mathcal{C}}_{\mathcal{B}}$

- la composizione di appl. lin. si traduce nel prodotto righe x col. delle matrici associate (se uso le stesse basi)