# Ist. Mat. I - C1A

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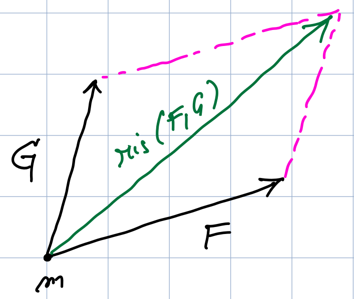
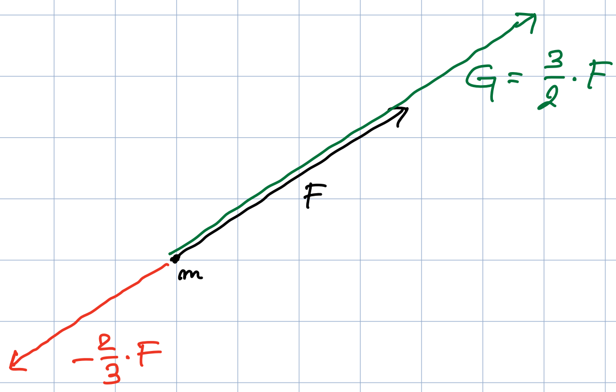
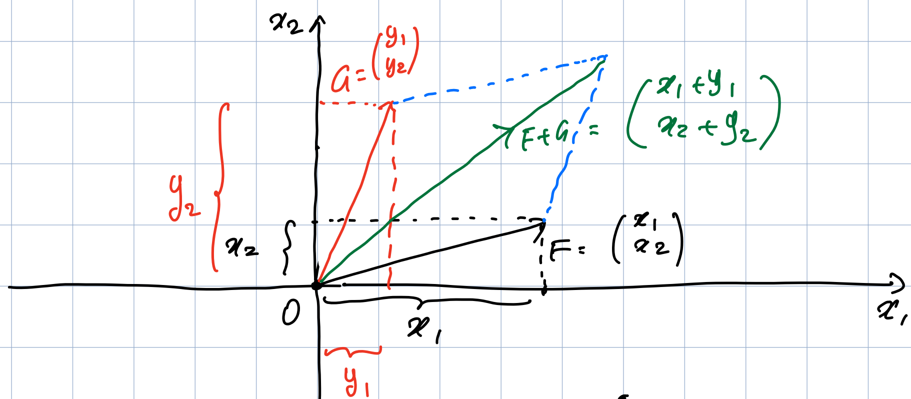
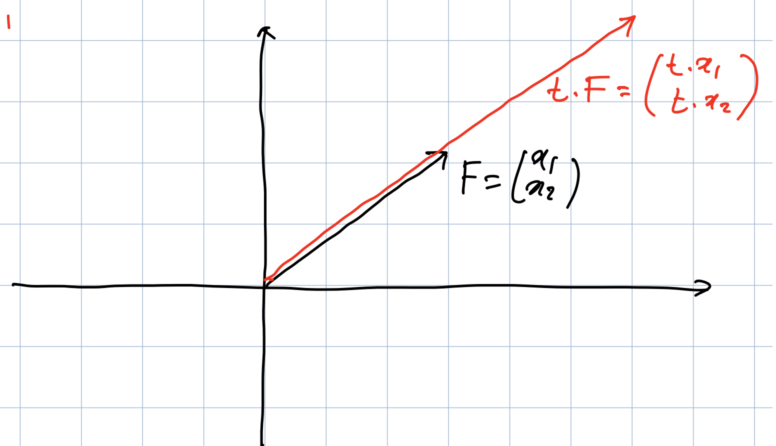



ris(F, G) = F + G



$$\mathbb{R}^2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1, x_2 \in \mathbb{R} \right\}$$

- $\mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

  $(x, y) \longmapsto x + y$

  $$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$$

- $\mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

  $(t, x) \longmapsto t.x$

  $$t.\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t.x_1 \\ t.x_2 \end{pmatrix}$$

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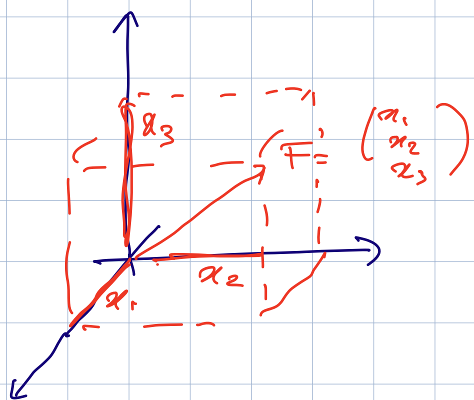
$\mathbb{R}^3$




$$\underset{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}{\overset{\|}{F}} + \underset{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}{\overset{\|}{G}} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix}$$

$$t.\underset{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}{\overset{\|}{F}} = \begin{pmatrix} t.x_1 \\ t.x_2 \\ t.x_3 \end{pmatrix}$$

$n \in \mathbb{N}$ $\quad \mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : x_1, \dots, x_n \in \mathbb{R} \right\}$

- $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$
  $(x, y) \longmapsto x+y$
  $$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{pmatrix}$$
- $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$
  $(t, x) \longmapsto t \cdot x$
  $$t \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} t \cdot x_1 \\ t \cdot x_2 \\ \vdots \\ t \cdot x_n \end{pmatrix}$$

<u>Notazione</u> : se $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ indico $x_j$ con $(x)_j$

<u>Es</u>: se $x = \begin{pmatrix} \sqrt{3} \\ -7 \\ 4 \\ 2e \end{pmatrix} \in \mathbb{R}^4$

$$(x)_3 = 4$$

Le operazioni dunque sono definite da:

$x, y \in \mathbb{R}^n$; $(x+y)_j = (x)_j + (y)_j$ ; $t \in \mathbb{R}$, $x \in \mathbb{R}^n$
$(t \cdot x)_j = t \cdot (x_j)$

———— o ————

Proprietà:

1. se $0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ ho $0 + x = x$ $\quad \forall x \in \mathbb{R}^n$
2. se $-x = \begin{pmatrix} -x_1 \\ \vdots \\ -x_n \end{pmatrix}$ ho $(-x) + x = 0$ $\quad \forall x \in \mathbb{R}^n$
3. $x + (y+z) = (x+y) + z$ $\quad \forall x, y, z \in \mathbb{R}^n$
4. $x + y = y + x$ $\quad \forall x, y \in \mathbb{R}^n$

5. $t\cdot(s\cdot x) = (t\cdot s)\cdot x \qquad \forall t,s\in\mathbb{R} \quad \forall x\in\mathbb{R}^n$

6. $t\cdot(x+y) = \cancel{(}t\cdot x\cancel{)}+\cancel{(}t\cdot y\cancel{)}$

7. $(t+s)\cdot x = \cancel{(}t\cdot x\cancel{)}+\cancel{(}s\cdot x\cancel{)}$

il · si esegue prima del +

8. $1\cdot x = x$

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**Def:** chiamo spazio vettoriale su $\mathbb{R}$ un insieme $V$ con due operazioni

$$V\times V \longrightarrow V \qquad\qquad \mathbb{R}\times V \longrightarrow V$$
$$(v,w)\longmapsto v+w \qquad\qquad (t,v)\longmapsto t\cdot v$$

con:

1. $\exists\, 0\in V$ t.c. $0+v=v \quad \forall v$
2. $\forall v\ \exists\,(-v)$ t.c. $(-v)+v=0$
3. $v+(w+z) = (v+w)+z$
4. $v+w = w+v$
5. $(t\cdot s)\cdot v = t\cdot(s\cdot v)$
6. $t\cdot(v+w) = t\cdot v + t\cdot w$
7. $(t+s)\cdot v = t\cdot v + s\cdot v$
8. $1\cdot v = v$

**Es:** $\mathcal{M}_{m\times n}(\mathbb{R}) = \left\{ \underbrace{\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{pmatrix}}_{n} \Big\}m \ : \ a_{ij}\in\mathbb{R} \right\}$

$\parallel$

matrici $m\times n$

righe $\quad$ colonne

$$A = \begin{pmatrix} \sqrt{2} & -7 & 4\pi \\ 11 & -73/4 & e \end{pmatrix} \in \mathcal{M}_{2\times 3}(\mathbb{R})$$

<u>Notazione</u>: se $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$

indico $a_{ij}$ con $(A)_{ij}$

indice di riga

indice di colonna

$$(A)_{22} = -\frac{73}{4} \qquad (A)_{13} = 4\pi$$

Operazioni di sp. vett.:

$$(\underbrace{\underbrace{A}_{m\times n} + \underbrace{B}_{m\times n}}_{m\times n})_{ij} = (A)_{ij} + (B)_{ij}$$

$$\begin{pmatrix} 7 & 4 & -5 \\ 2 & 1 & 3 \end{pmatrix} + \begin{pmatrix} -3 & 1 & -2 \\ 4 & 7 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 5 & -7 \\ 6 & 8 & 14 \end{pmatrix}$$

$$(\underbrace{t \cdot \underbrace{A}_{m\times n}}_{m\times n})_{ij} = t \cdot (A)_{ij}$$

$$2 \cdot \begin{pmatrix} 5/2 & -7 & 11 \\ 4 & -19/4 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -14 & 22 \\ 8 & -19/2 & 6 \end{pmatrix}$$

<u>Fatto</u>: valgono le proprietà 1–8 dove

$$O = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix}$$

$$-\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{pmatrix} = \begin{pmatrix} -a_{11} & \dots & -a_{1n} \\ \vdots & & \vdots \\ -a_{m1} & \dots & -a_{mn} \end{pmatrix}$$

<u>Oss</u>: $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n = \mathcal{M}_{n\times 1}(\mathbb{R})$

$$(x_1 \dots x_n) \in \mathcal{M}_{1\times n}(\mathbb{R})$$

<u>Es</u>: $\mathbb{R}[t]$ = l'insieme dei polinomi a coeff. reali nella indeterminata $t$.

E' sp. vett. con le operazioni solite.

$$(1+\sqrt{3}\cdot t - 5t^2) + (2 + 6t^2 - t^4)$$

$$= 3 + \sqrt{3}t + t^2 - t^4$$

$$12 \cdot \left( \frac{1}{3} - \frac{7}{4} t + 9 t^7 \right) = 4 - 21t + 108t^7$$

Fatto: valgono tutte le proprietà 1-8.

Oss: è definito anche

$$\mathbb{R}[t] \times \mathbb{R}[t] \longrightarrow \mathbb{R}[t]$$
$$(p(t), q(t)) \longmapsto p(t) \cdot q(t)$$

però per la def. di sp. vett. serve solo

$$\mathbb{R} \times \mathbb{R}[t] \longrightarrow \mathbb{R}[t]$$
$$(\alpha, p(t)) \longmapsto \alpha \cdot p(t).$$

Oss: in uno sp. vett. $V$ si ha $0 \cdot v = 0$

$$\boxed{0} \underset{2}{=} (-0 \cdot v) + (0 \cdot v) = (-0 \cdot v) + ((0+0) \cdot v)$$
$$\underset{7}{=} (-0 \cdot v) + ((0 \cdot v) + (0 \cdot v))$$
$$\underset{4}{=} ((-0 \cdot v) + 0 \cdot v) + (0 \cdot v)$$
$$\underset{2}{=} 0 + (0 \cdot v)$$
$$\underset{1}{=} \boxed{0 \cdot v}$$

Fatto: tutte le proprietà algebriche naturali sono vere.

Estensione: uno spazio vett. su $\mathbb{C}$ è un insieme $V$ con operazioni:

$$V \times V \to V \qquad \mathbb{C} \times V \to V$$
$$(v, w) \mapsto v + w \qquad (\alpha, v) \mapsto \alpha \cdot v$$

con le 8 proprietà di sopra

Es: $\mathbb{C}^2 = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} : z_1, z_2 \in \mathbb{C} \right\}$

$$\begin{pmatrix} 7 - i \\ 2 + 5i \end{pmatrix} + \begin{pmatrix} 4 - 7i \\ 1 + 9i \end{pmatrix} = \begin{pmatrix} 11 - 8i \\ 3 + 14i \end{pmatrix}$$

$$(1+2i) \cdot \begin{pmatrix} 1 - 5i \\ 2 + 3i \end{pmatrix} = \begin{pmatrix} 11 - 3i \\ -4 + 8i \end{pmatrix}$$

———— o ————

Fisso sp. vett. $V$.

Oss: se $\alpha_1, \dots, \alpha_m \in \mathbb{R}$, $v_1, \dots, v_m \in \mathbb{R}$ ho

$$\underbrace{\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \dots + \alpha_m \cdot v_m}_{} \in V$$

combinazione lineare dei vettori $v_1, \dots, v_m$ con coeff. $\alpha_1, \dots, \alpha_m$

sia l'espressione
sia il suo risultato.

$3 + 5 \longleftrightarrow 8$

Oss: la combinazione lineare con coeff. tutti 0 dà risultato 0

Dati $v_1, \dots, v_m \in V$ posso chiedermi:

(i) • l'unico modo per avere risultato 0 è tutti coeff. 0?

(ii) • posso esprimere tutti i vettori di $V$ come loro comb. lin.?

(iii) • in tal caso, in modo unico?

Esempi in $\mathbb{R}^2$: $\underline{m=1}$

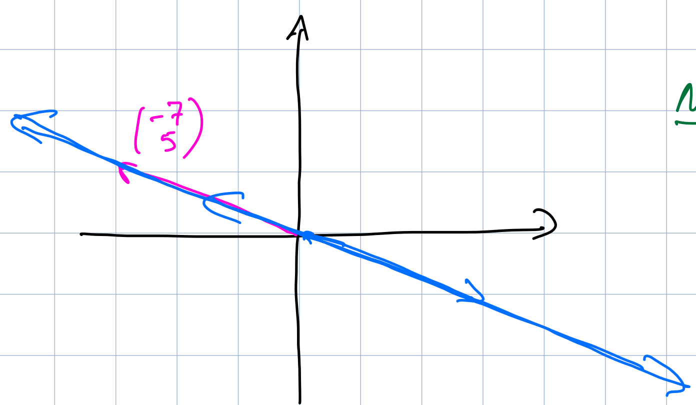
• $v = \begin{pmatrix} -7 \\ 5 \end{pmatrix}$

(i) $\alpha \cdot \begin{pmatrix} -7 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \alpha = 0$ sì

(ii) dato $v \in \mathbb{R}^2$ qualsiasi esiste $\alpha$ t.c. $\alpha \cdot \begin{pmatrix} -7 \\ 5 \end{pmatrix} = v$?

No: ad es. se $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\alpha \cdot \begin{pmatrix} -7 \\ 5 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \forall \alpha$




• $v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(i) $\alpha \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \not\Rightarrow \alpha = 0$ No

(ii) No

$\underline{m=2}$

• $v_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, v_2 = \begin{pmatrix} -2 \\ 7 \end{pmatrix}$

(i) $\alpha_1 \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} -2 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\overset{?}{\Longrightarrow} \alpha_1 = \alpha_2 = 0$

$\Updownarrow$

$$\begin{pmatrix} \alpha_1 - 2\alpha_2 \\ 4\alpha_1 + 7\alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Updownarrow$

$$\begin{cases} \alpha_1 - 2\alpha_2 = 0 \\ 4\alpha_1 + 7\alpha_2 = 0 \end{cases}$$

$\Updownarrow$

$$\begin{cases} \alpha_1 = 2\alpha_2 \\ 8\alpha_2 + 7\alpha_2 = 0 \end{cases}$$

$\Updownarrow$

$$\begin{cases} 15\alpha_2 = 0 \\ \alpha_1 = 2\alpha_2 \end{cases} \Longleftrightarrow \begin{cases} \alpha_2 = 0 \\ \alpha_1 = 0 \end{cases}$$

Sì

(ii) e (iii) dato $x \in \mathbb{R}^2$ qualsiasi esistono $\alpha_1, \alpha_2$ t.c.

$\alpha_1 \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} -2 \\ 7 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ ? Unici?

$\Updownarrow$

$$\begin{cases} \alpha_1 - 2\alpha_2 = x_1 \\ 4\alpha_1 + 7\alpha_2 = x_2 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} \alpha_1 = 2\alpha_2 + x_1 \\ 8\alpha_2 + 4x_1 + 7\alpha_2 = x_2 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} 15\alpha_2 = -4x_1 + x_2 \\ \alpha_1 = 2\alpha_2 + x_1 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} \alpha_2 = \dfrac{-4x_1 + x_2}{15} \\ \alpha_1 = \dfrac{-8x_1 + 2x_2}{15} + x_1 = \dfrac{7x_1 + 2x_2}{15} \end{cases}$$

Sì, sì

- $v_1 = \begin{pmatrix} 8 \\ -6 \end{pmatrix}, v_2 = \begin{pmatrix} -20 \\ 15 \end{pmatrix}$

(i) $\alpha_1 \begin{pmatrix} 8 \\ -6 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} -20 \\ 15 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Leftrightarrow \begin{cases} 8\alpha_1 - 20\alpha_2 = 0 \\ -6\alpha_1 + 15\alpha_2 = 0 \end{cases} \Leftrightarrow \begin{cases} 2\alpha_1 - 5\alpha_2 = 0 \\ 2\alpha_1 - 5\alpha_2 = 0 \end{cases}$$

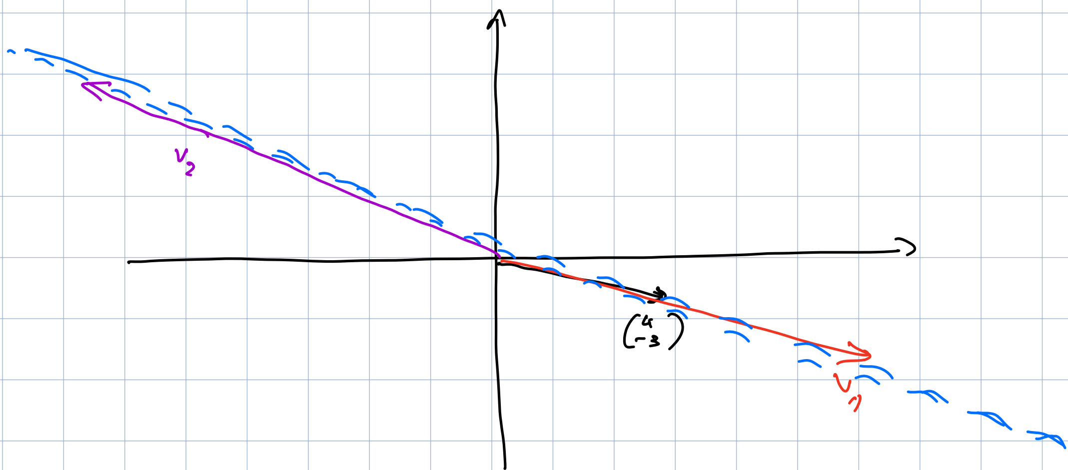
$\Leftrightarrow 2\alpha_1 - 5\alpha_2 = 0$ non ha solo la soluz. $\alpha_1 = \alpha_2 = 0$

ad es. $\alpha_1 = 5 \quad \alpha_2 = 2$

No

Oss: $v_1 = 2 \cdot \begin{pmatrix} 4 \\ -3 \end{pmatrix} \qquad v_2 = (-5) \cdot \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

(ii) dato $x \in \mathbb{R}^2$ esistono sempre $\alpha_1, \alpha_2$ t.c.
$$\alpha_1 \cdot \begin{pmatrix} 8 \\ -6 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} -20 \\ 15 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} ?$$




No: si trovano solo gli $x$ multipli di $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ con tutti

$n=3$ $\quad v_1 = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \quad v_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad v_3 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

(i) $\alpha_1 \cdot \begin{pmatrix} -1 \\ 4 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 2 \\ 5 \end{pmatrix} + \alpha_3 \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} = 0 \overset{?}{\Longrightarrow} \alpha_1 = \alpha_2 = \alpha_3 = 0$

$$\Updownarrow$$

$$\begin{cases} -\alpha_1 + 2\alpha_2 - 2\alpha_3 = 0 \\ 4\alpha_1 + 5\alpha_2 + \alpha_3 = 0 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} \alpha_1 = 2\alpha_2 - 2\alpha_3 \\ 8\alpha_2 - 8\alpha_3 + 5\alpha_2 + \alpha_3 = 0 \end{cases} \qquad \begin{cases} 13\alpha_2 - 7\alpha_3 = 0 \\ \alpha_1 = 2\alpha_2 - 2\alpha_3 \end{cases}$$

No: ad es $\alpha_2 = 7 \quad \alpha_3 = 13$
$\alpha_1 = -12$ $\qquad$ No

(ii) dato $x \in \mathbb{R}^2$ esistono sempre $\alpha_1, \alpha_2, \alpha_3$ t.c.

$$\alpha_1 \cdot \begin{pmatrix} -1 \\ 4 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 2 \\ 5 \end{pmatrix} + \alpha_3 \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} ?$$

Unici ?

$$\Updownarrow$$

$$\begin{cases} -\alpha_1 + 2\alpha_2 - 2\alpha_3 = x_1 \\ 4\alpha_1 + 5\alpha_2 + \alpha_3 = x_2 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} \alpha_1 = 2\alpha_2 - 2\alpha_3 - x_1 \\ 8\alpha_2 - 8\alpha_3 - 4x_1 + 5\alpha_2 + \alpha_3 = x_2 \end{cases}$$

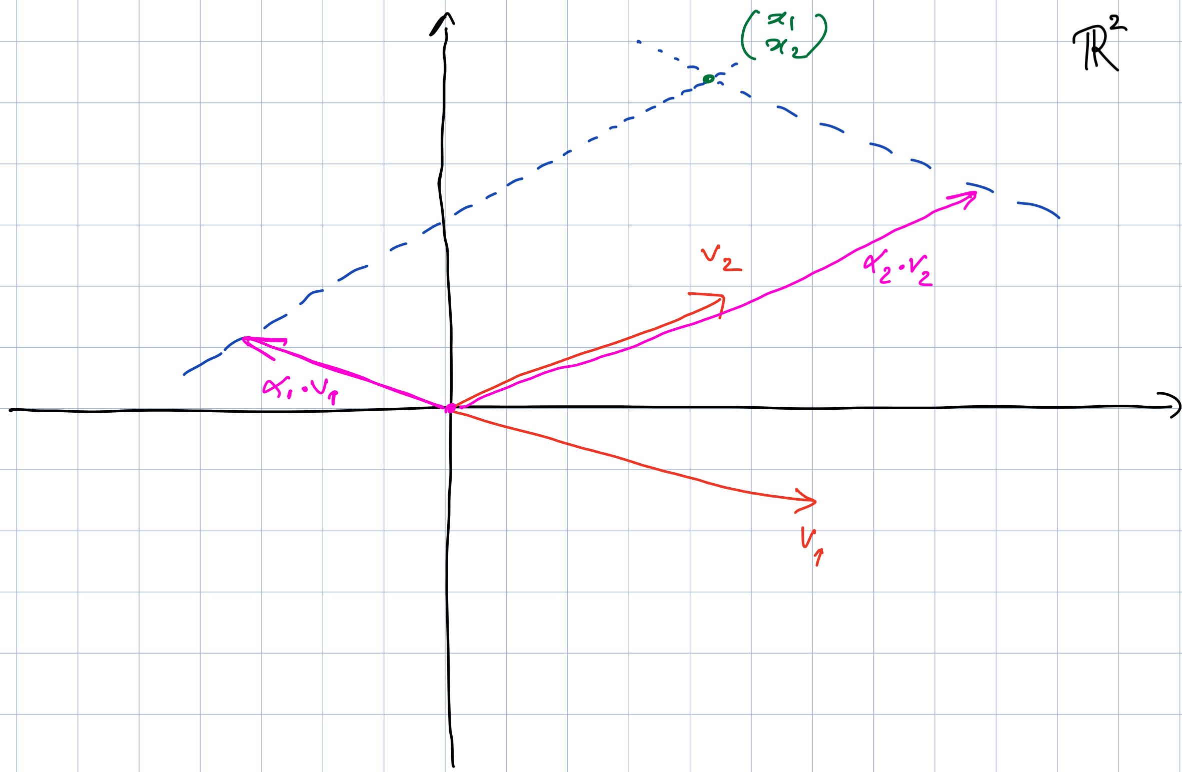
$$\Updownarrow$$

$$\begin{cases} 13\alpha_2 - 7\alpha_3 = 4x_1 + x_2 \\ \alpha_1 = 2\alpha_2 - 2\alpha_3 - x_1 \end{cases}$$

Ha soluzioni non uniche:

- $\alpha_3 = 0$ , $\alpha_2 = \dfrac{4x_1 + x_2}{13}$ , $\alpha_1 = \dots$
- $\alpha_2 = 0$ , $\alpha_3 = -\dfrac{4x_1 + x_2}{7}$ , $\alpha_1 = \dots$

<u>Sì</u>, no


$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$
$\mathbb{R}^2$
$v_2$
$\alpha_2 \cdot v_2$
$\alpha_1 \cdot v_1$
$v_1$