Ist. Mat. I - C/A
30/11/23

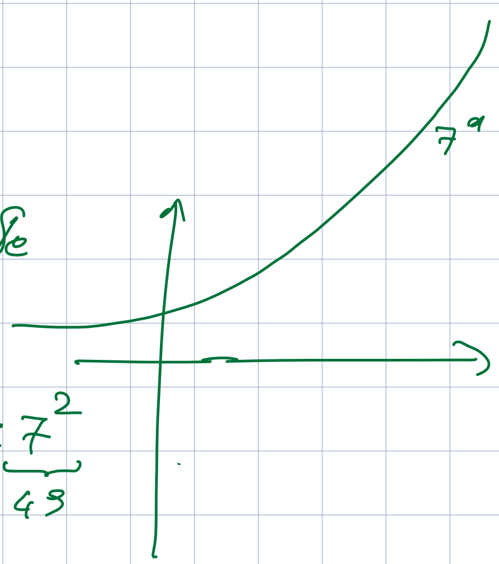
$\boxed{31/1/23}$

(1) Noto: $\sqrt{2} \notin \mathbb{Q}$ provare che $\underbrace{\underbrace{4.7\overline{13}}_{a}+\sqrt{2}}_{b} \notin \mathbb{Q}$

$a \in \mathbb{Q}$; se per assurdo $b \in \mathbb{Q}$ allora $b-a \in \mathbb{Q}$
ma $b-a = \sqrt{2}$: assurdo.

(2) $\log_7(48) < 2$

So che $x \longmapsto 7^x$ è crescente
quindi:

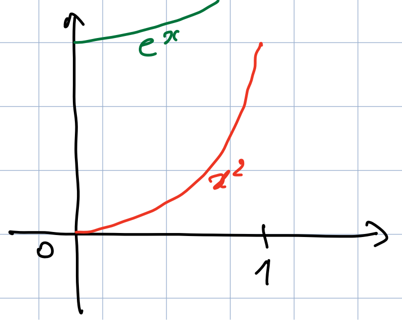


$$\log_7(48) < 2 \iff \underbrace{7^{\log_7(48)}}_{48} < \underbrace{7^2}_{49}$$

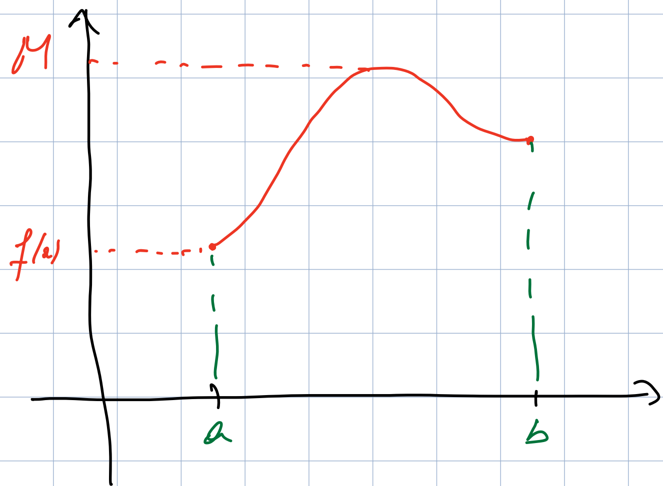
vero

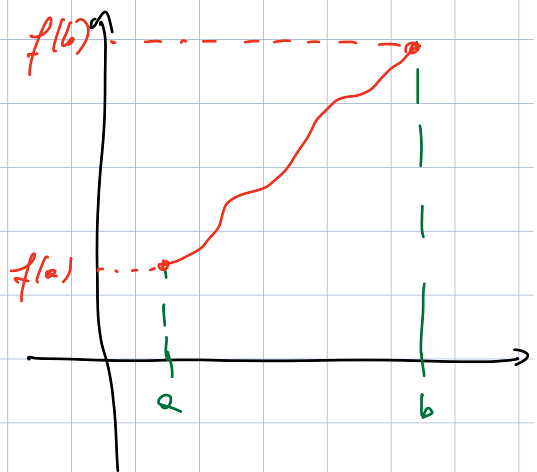
(3) $$\lim_{n\to\infty} \frac{3n^2+(-1)^n \cdot n}{2n^2+\sin(n \cdot \frac{\pi}{2})} = \frac{3}{2}$$

(4) $f:[0,1] \to \mathbb{R}$ $\quad f(x) = e^x + x^2$
$J = \mathrm{Im}(f)$; $f: I \to J$ invertibile con inv. cont.?



f in $[0,1]$ è $\nearrow + \nearrow \Rightarrow f \nearrow$
$\Rightarrow J = [f(0), f(1)] = [1, 1+e]$.
Essendo $\nearrow$ f è invertibile;
f continua $\Rightarrow f^{-1}$ continua.





$$\left(f'(x) = e^x + 2x > 0 \quad \text{in } [0,1] \right)$$

(5) $f(x) = \sin(\sqrt{x})$. D insieme su cui è definita.
Trovare $f'(x)$ e dire dove è definita.

$$D = [0, +\infty)$$

$$f'(x) = \cos(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} \qquad \text{su } (0, +\infty)$$

(6) $$\lim_{x \to 0} \frac{1-\cos(x)}{x \cdot \log(1+x)} \qquad \frac{0}{0}$$

$$\frac{1-\cos(x)}{x \cdot \log(1+x)} = \underbrace{\underbrace{\frac{1-\cos(x)}{x^2}}_{\frac{1}{2}} \cdot \underbrace{\left(\underbrace{\frac{\log(1+x)}{x}}_{1} \right)^{-1}}_{1}}_{\frac{1}{2}}$$

• $\exists \quad \dfrac{1-\left(1-\frac{1}{2}x^2+o(x^2)\right)}{x\cdot\left(x+o(x)\right)} = \dfrac{\frac{1}{2}x^2+o(x^2)}{x^2+o(x^2)} \to \dfrac{1}{2}$

⑦ Taylor II in $x=0$ per $f(x)=e^{x-x^2}$.

$$\bullet\quad e^{x-x^2} = 1+(x-x^2)+\frac{1}{2}(x-x^2)^2+o\left((x-x^2)^2\right)$$

$$= 1+(x-x^2)+\frac{1}{2}x^2+o(x^2)$$

$$= 1+x-\frac{1}{2}x^2+o(x^2)$$

$\bullet\quad f(x)=e^{x-x^2}$ $\qquad f(0)=1$

$f'(x)=e^{x-x^2}(1-2x)$ $\qquad f'(0)=1$

$f''(x)=e^{x-x^2}(1-2x)^2+e^{x-x^2}\cdot(-2)$ $\qquad f''(0)=1-2=-1$

$$f(x)=f(0)+f'(0)\cdot x+\frac{1}{2}f''(0)\cdot x^2+o(x^2)$$

$$=1+x-\frac{1}{2}x^2+o(x^2)$$

⑧ Provare che $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\log(n)}$ converge.

Leibniz: $\displaystyle\sum_{n=0}^{\infty}(-1)^n\cdot a_n \qquad a_n \searrow 0 \Rightarrow$ converge

$\log(n) \nearrow +\infty \Rightarrow \dfrac{1}{1+\log(n)} \searrow 0$

$\boxed{7/6/23}$

1. Provare per induzione che $6^n - (-1)^n$ è multiplo di 7 $\forall n \in \mathbb{N}$

Passo base: $n = 0$ $\quad 6^0 - (-1)^0 = 1 - 1 = 0$ ✓

($n = 1$ $\quad 6^1 - (-1)^1 = 6 - (-1) = 7$ ✓)

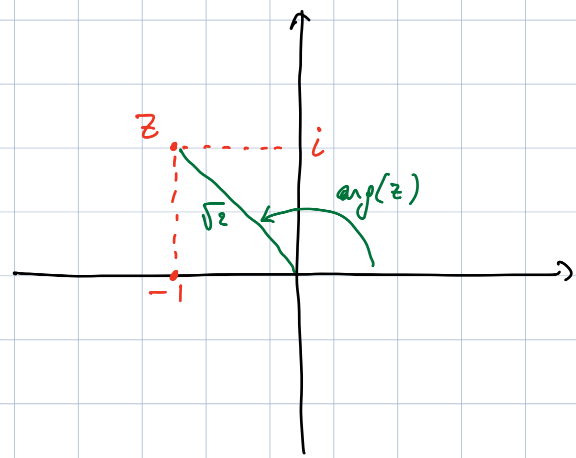
Passo induttivo: suppongo $6^n - (-1)^n$ multiplo di 7
dimostro $6^{n+1} - (-1)^{n+1}$ multiplo di 7.

Ipotesi: $6^n - (-1)^n = 7k \qquad k \in \mathbb{N}$

$$
\begin{aligned}
6^{n+1} - (-1)^{n+1} &= 6 \cdot 6^n + (-1)^n \\
&= 6(7k + (-1)^n) + (-1)^n \\
&= 6 \cdot 7k + 6 \cdot (-1)^n + (-1)^n \\
&= 7 \cdot 6k + 7 \cdot (-1)^n \\
&= 7 \cdot \underbrace{(6k + (-1)^n)}_{\in \mathbb{N}}
\end{aligned}
$$

$\Rightarrow$ ok

② $z = -1+i$. $|z|$, $\arg(z)$



$|z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\arg(z) = \frac{3}{4}\pi + 2k\pi$

$k \in \mathbb{Z}$

③ $$\lim_{n \to +\infty} \frac{3n + (-1)^n \cdot n^2}{\underbrace{\cos(n\pi)}_{(-1)^n} - 5n^2}$$

$\nexists$

④ $f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = 2 \cdot \cos(x) - 3x$.

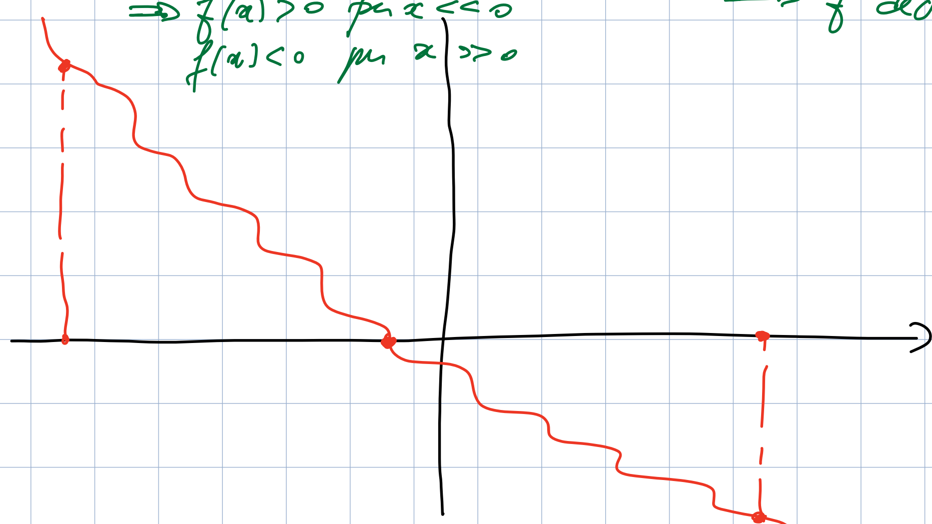
Provare che ha uno e un solo zero.

$\lim_{x \to \pm\infty} f(x) = \mp\infty$; $f'(x) = -2 \cdot \sin(x) - 3 < 0$

$\Rightarrow f(x) > 0$ per $x << 0$

$f(x) < 0$ per $x >> 0$

$\Rightarrow f$ decrescente



⑤ $\underbrace{\text{Ordine di zero}}$ in $x=0$ di $f(x) = x^2 \cdot \sin(x^3) \cdot \log(1+x)$

- $k \in \mathbb{N}$ t.c. $\lim_{x\to 0} \frac{f(x)}{x^k} = \alpha$ esiste finito non nullo
- $f(x) = \alpha \cdot x^k + o(x^k) \qquad \alpha \neq 0$

$$f(x) = x^2 \cdot \left(x^3 + o(x^3)\right) \cdot \left(x + o(x)\right)$$
$$= x^6 + o(x^6)$$

$$\boxed{k=6}$$

⑥ Lagrange applicato a $f: [0,2] \to \mathbb{R}$ $\quad f(x) = \frac{\log_3(1+x^3)}{1+x^2}$
fornisce $x \in (0,2)$ con $f'(x) = ?$

Lagrange: $f: [a,b] \to \mathbb{R}$ continua, derivabile su (a,b)
$\Rightarrow \exists x \in (a,b)$ con $f'(x) = \frac{f(b)-f(a)}{b-a}$

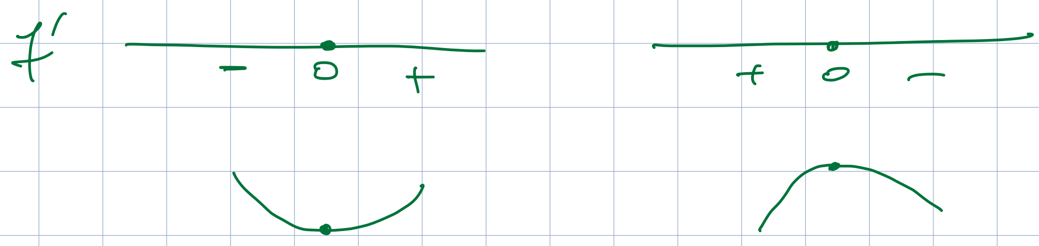
$$\exists x \text{ con } f'(x) = \frac{\frac{\log_3(1+8)}{1+4} - \frac{\log_3(1+0)}{1+0}}{2-0}$$

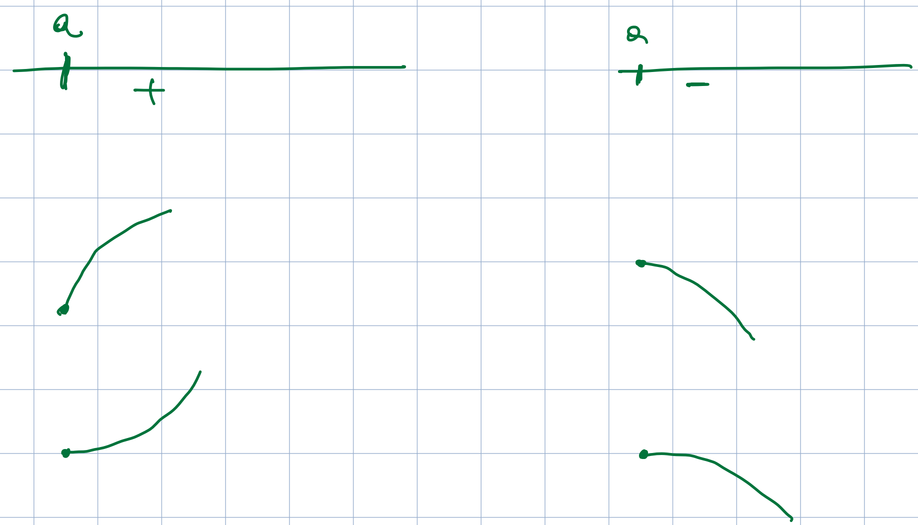
$$= \frac{\frac{2}{5} - 0}{2-0} = \frac{1}{5}$$

Come decidere se anche gli estremi del dominio di una f sono max/min loc.

$f : [a,b] \to \mathbb{R}$ continua e derivabile

Per trovare max/min loc calcolo $f'(x)$ e studio segno.





7. Taylor III in $x=0$ per $f(x)=e^x\cdot\cos(x)$

- $f(x)=\left(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+o(x^3)\right)\cdot\left(1-\frac{1}{2}x^2+o(x^3)\right)$

$$=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+o(x^3)$$
$$-\frac{1}{2}x^2-\frac{1}{2}x^3+o(x^3)$$
$$+o(x^3)$$

$$=1+x-\frac{1}{3}x^3+o(x^3)$$

(Taylor III per $\frac{1}{x}(1-\cos(x))$)

$$\cancel{\frac{1}{x}\left(1-\left(1-\frac{1}{2}x^2+o(x^3)\right)\right)}$$

$$\frac{1}{x}\left(1-\left(1-\frac{1}{2}x^2+\frac{1}{24}x^4+o(x^5)\right)\right)$$

$$=\frac{1}{2}x-\frac{1}{24}x^3+o(x^4)$$

- $f(x)=e^x\cdot\cos(x)$ $\qquad f(0)=1$

$f'(x)=e^x\cdot(\cos(x)-\sin(x))$ $\qquad f'(0)=1$

$f''(x)=e^x\cdot(-2\sin(x))$ $\qquad f''(0)=0$

$f'''(x)=e^x(-2\sin(x)-2\cos(x))$ $\qquad f'''(0)=-2$

$$f(0)+f'(0)\cdot x+\frac{f''(0)}{2}\cdot x^2+\frac{f'''(0)}{6}\cdot x^3+o(x^3)$$

$$1+x-\frac{1}{3}x^3+o(x^3)$$

8. Calcolare $\displaystyle\sum_{n=1}^{\infty}\sin\left(\frac{1}{n}\right)-\sin\left(\frac{1}{n+1}\right)$

$$S_k=\sum_{n=1}^{k}(\dots)=\left(\sin(1)-\cancel{\sin\left(\frac{1}{2}\right)}\right)+$$

$$\left(\cancel{\sin\left(\tfrac{1}{2}\right)} - \cancel{\sin\left(\tfrac{1}{3}\right)}\right) +$$

$$\left(\cancel{\sin\left(\tfrac{1}{3}\right)} - \cancel{\sin\left(\tfrac{1}{4}\right)}\right) +$$

$$\dots$$

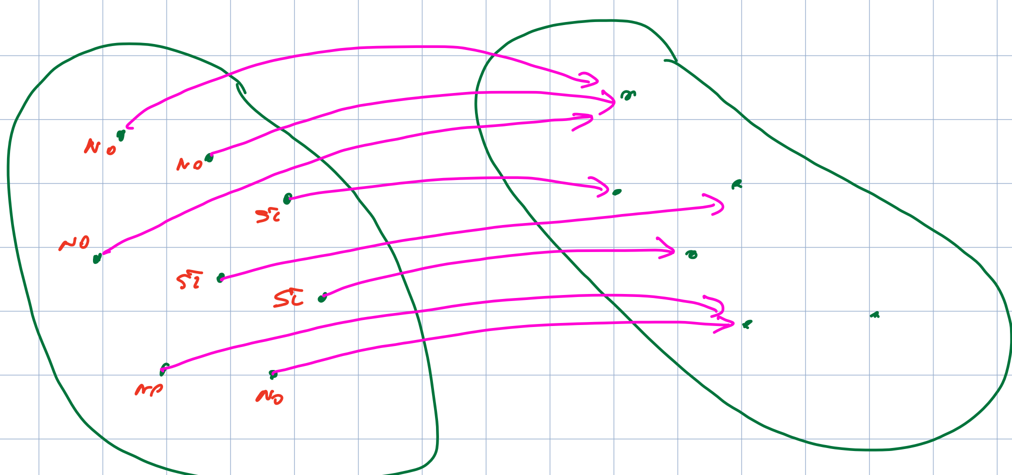
$$\cancel{\sin\left(\tfrac{1}{k}\right)} - \sin\left(\tfrac{1}{k+1}\right)$$

$$= \underbrace{\sin(1) - \underbrace{\sin\Big(\underbrace{\tfrac{1}{k+1}}_{\to 0}\Big)}_{\to 0}}_{\to \sin(1)}$$

$\boxed{27/6/23}$ ① $X = \{n \in \mathbb{N} : n \le 10\}$ $Y =$ alfabeto

$f : X \to Y$ $f(n) =$ seconda lettera di n in italiano

$\{x \in X : f^{-1}(f(x)) = \{x\}\}.$



NO	SÌ	NO	SÌ	NO	NO	NO	NO	SÌ	SÌ	NO
0	1	2	3	4	5	6	7	8	9	10
E	N	U	R	U	I	E	E	T	O	I

$\{1, 3, 8, 9\}$

② $$\frac{\log(3)}{\log(2)} \cdot \frac{\log_3(6)}{\log_2(6)}$$

$$= \frac{\log(3)}{\log(2)} \cdot \frac{\log_3(2) + \log_3(3)}{\log_2(2) + \log_2(3)}$$

$$= \frac{\log(3)}{\log(2)} \cdot \frac{\frac{\log(2)}{\log(3)} + 1}{1 + \frac{\log(3)}{\log(2)}} = \frac{\cancel{\log(3)}}{\cancel{\log(2)}} \cdot \frac{\frac{\cancel{\log(2)+\log(3)}}{\cancel{\log(3)}}}{\frac{\cancel{\log(2)+\log(3)}}{\cancel{\log(2)}}}$$

$$= 1$$

$$\log_a(b) = \frac{\log_c(b)}{\log_c(a)}$$

③ $a_n = \sqrt{n^2 - 1}$ $\qquad b_n = a_{n+1} - a_n$ $\qquad \lim_{n \to \infty} b_n = ?$

$$b_n = \sqrt{n^2 + 2n} - \sqrt{n^2 - 1} = \frac{\cancel{n^2} + 2n - \cancel{n^2} + 1}{\sqrt{n^2 + 2n} + \sqrt{n^2 - 1}} \to 1$$

④ $$\lim_{x \to \infty} \frac{1 - \cos(e^{-x})}{e^{-2x}} \qquad \frac{0}{0}$$

$t = e^{-x}$ $\qquad x \to +\infty \Rightarrow t \to 0$

$$\frac{1-\cos(e^{-x})}{e^{-2x}} = \frac{1-\cos(t)}{t^2} \longrightarrow \frac{1}{2}$$

5 $f(x) = \arcsin(\sqrt{x})$ dominio

$f'(x) = ?$ dominio

$$D: \begin{cases} x \geq 0 \\ |\sqrt{x}| \leq 1 \end{cases} \qquad D = [0,1]$$

$$f'(x) = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}} \qquad \lim = +\infty \text{ in } 0 \text{ e in } 1$$

$\Rightarrow f'$ esiste su $(0,1)$

6 $$\lim_{x \to 0} \frac{1-e^x + \log(1+x)}{x^2} \qquad \frac{1-1+0}{0} = \frac{0}{0}$$

- $$\overset{?}{=} \lim_{x\to 0} \frac{1-\left(1+x+\frac{1}{2}x^2+o(x^2)\right)+\left(x-\frac{1}{2}x^2+o(x^2)\right)}{x^2}$$

$$= \lim_{x\to 0} \frac{-x^2+o(x^2)}{x^2} = -1$$

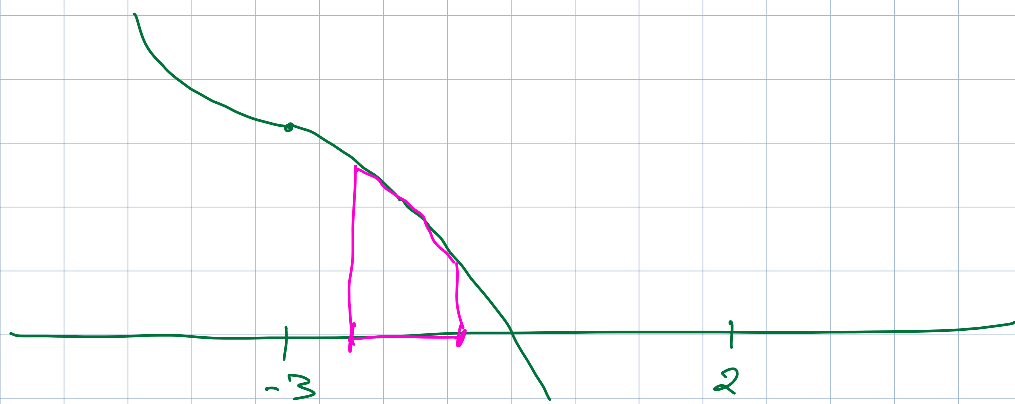
- $$\overset{?}{=} \lim \frac{-e^x + \frac{1}{1+x}}{2x} \overset{?}{=} \lim \frac{-e^x - \frac{1}{(1+x)^2}}{2} = \frac{-1-1}{2} = -1$$

7 $f(x) = x^4 + 2x^3 - 36x^2$ intervalli di conc/conv.

$$f'(x) = 4x^3 + 6x^2 - 72x$$

$$f''(x) = 12x^2 + 12x - 72$$
$$= 12(x^2 + x - 6)$$
$$= 12(x+3)(x-2)$$

convesse sugli intervalli $(-\infty, 3]$ e $[2, +\infty)$
concave sull'intervallo $[-3, 2]$



(8) $f : [1,2] \to \mathbb{R}$ $\quad f(x) = x^2 - \log(x) - 2$

Provare che si applica il metodo iterativo (con le tangenti) per la ricerca di zero a partire da ...

- $f : [a,b] \to \mathbb{R}$ $\qquad f(a) \cdot f(b) < 0$

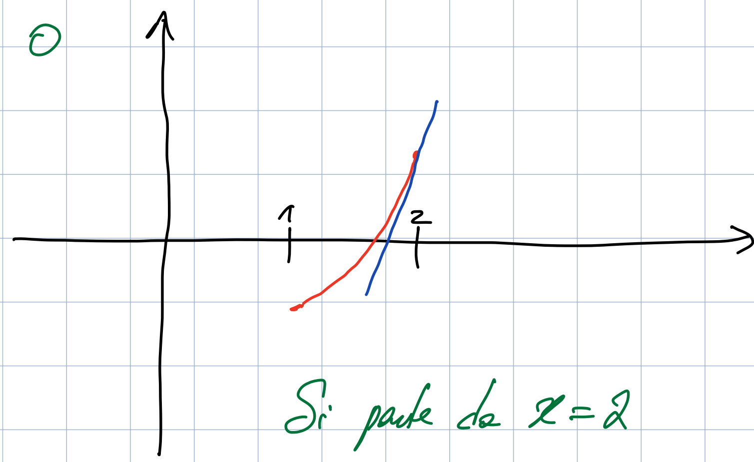
  f', f'' segno costante

$f(1) = 1 - 0 - 2 = -1$

$f(2) = 4 - \log(2) - 2 > 0$

$f'(x) = 2x - \frac{1}{x} > 0$

$f''(x) = 2 + \frac{1}{x^2} > 0$



Si parte da $x = 2$