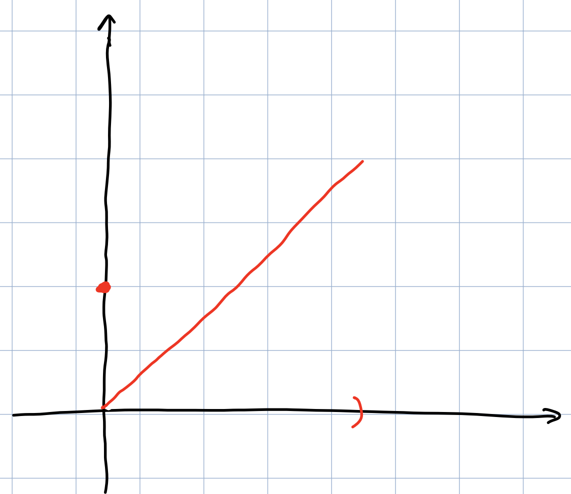
Ist. Mat. I - CIA
17/11/23

Foglio 5, Es 3 $\qquad \exists f^{-1}$ continua?

(e) $f : [0,2) \to \mathbb{R} \qquad f(x) = x+1 - \mathrm{sgn}(x)$



$f(0) = f(1)$ non iniettiva; inoltre discontinua in 0.

(f) $f : [-2,0] \to \mathbb{R} \qquad f(x) = x^3 - 2x$

Se su intervallo $I \subset \mathbb{R}$ la funz $x \mapsto x^3 - 2x$ è iniettiva allora è invertibile se abbreviato all'$\mathrm{Im}(f)$.

$f'(x) = 3x^2 - 2$ $\qquad$ cambia segno in $x = -\sqrt{2/3} \in (-2,0)$
$\Rightarrow$ NO.

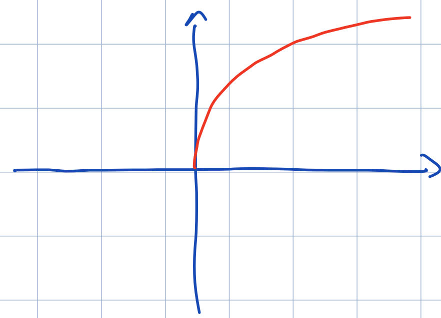
Zanichelli p.160

$\boxed{1}$ Che tipo di punto è 0 per $x \mapsto x^\alpha$
$\alpha = 1/3, 4/3, \ldots$

Oss: $x \mapsto x^{\alpha}$ è definito per $x \geq 0$ per α generale
Però ad esempio se $\alpha = 1/n$ n intero dispari
è definita anche per $x < 0$. Es: $\sqrt[3]{x}$ $\forall x \in \mathbb{R}$.

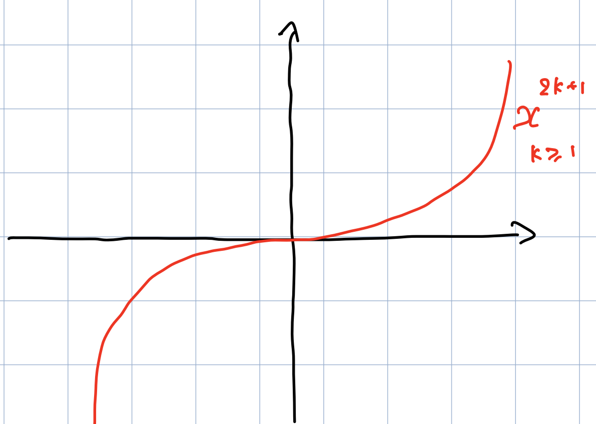
Att: molti software non lo sanno.

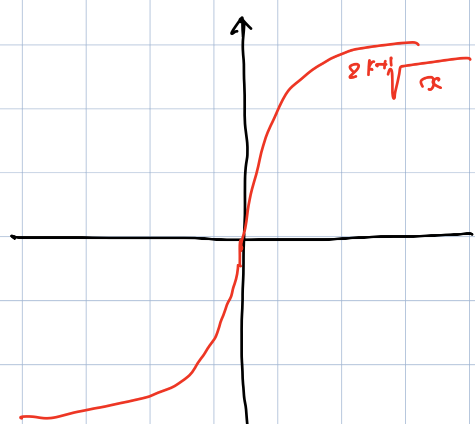
$\sqrt[3]{x} \rightsquigarrow$



$-\sqrt[3]{-x} \rightsquigarrow$

Oss: $x \longmapsto x^{1/2k+1} = \sqrt[2k+1]{x}$ è l'inversa di $x \longmapsto x^{2k+1}$





$\alpha > 0$

$f(x) = x^{\alpha}$ $\qquad$ $f'(x) = \alpha \cdot x^{\alpha-1}$

Se $\alpha < 1$ $\qquad$ $\lim_{x \to 0^+} f'(x) = +\infty$

$$f'_+(0) = \lim_{x \to 0} \frac{\alpha \cdot x^{\alpha} - 0}{x} = +\infty$$

0 punto a tangente verticale
(flesso se $\alpha = \frac{1}{2k+1}$)

Se $\alpha = 1$ ...

Se $\alpha > 1$ $\quad$ $f'(0) = 0$.

(2) $f(x) = \begin{cases} x \cdot \log(x) & x > 0 \\ 0 & x = 0 \end{cases}$ in $[0, +\infty)$

Verificare f continua ; calcolare $f'_+(0)$.

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \underbrace{x \cdot \log(x)}_{0 \cdot \infty} = 0 = f(0)$. OK continua.

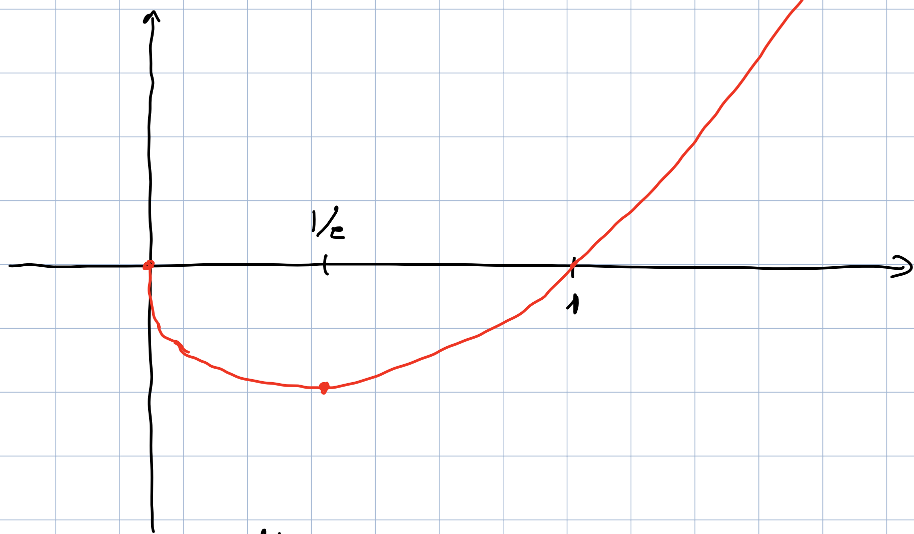
$$f'_+(0) = \lim_{x \to 0^+} \frac{x \cdot \log(x) - 0}{x} = \lim_{x \to 0^+} \log(x) = -\infty$$

$$f'(x) = 1 \cdot \log(x) + x \cdot \frac{1}{x} = \log(x) + 1 \longrightarrow -\infty$$

Oss: cambia segno in $x = e^{-1}$

$$\lim_{x \to +\infty} f(x) = +\infty$$

$f''(x) = \frac{1}{x}$ sempre >0



③ $f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x = 0 \end{cases}$ su $[0, +\infty)$

Continue in 0 + $f'_+(0)$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{-1/x} = e^{-\infty} = 0$$ ok

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x \to 0^+} \frac{1}{x} \cdot e^{-\frac{1}{x}} = \lim_{t \to +\infty} \underbrace{t \cdot e^{-t}}_{\infty \cdot 0} = 0$$

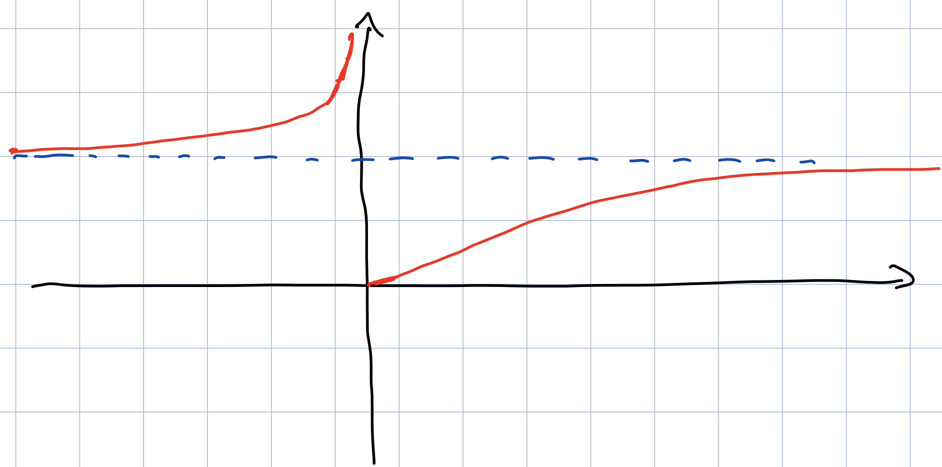
Ese: in realtà $\exists f_+^{(k)}(0) = 0 \quad \forall k$

poiché

$$f^{(k)}(x) = \frac{P_k(x)}{q_k(x)} \cdot e^{-1/x}$$ P_k, q_k polinomi.

$$\Longrightarrow f^{(k)}(x) \xrightarrow[x \to 0^+]{} 0$$

Oss: $e^{-1/x}$ su $\mathbb{R} \setminus \{0\}$



$$\left(e^{-1/x}\right)' = e^{-1/x} \cdot \frac{1}{x^2} > 0$$

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(4) $3x^4 + 5x + x^{3/2} - 2x^{-3}$

$$12x^3 + 5 + \frac{3}{2}x^{1/2} + 6x^{-4}$$

(5) $\log\left|\frac{x+2}{3-x}\right|$

- $$\frac{1}{\frac{x+2}{\cancel{3-x}}} \cdot \frac{1\cdot(3-x) - (-1)\cdot(x+2)}{(3-x)^{\cancel{2}}} = \frac{5}{(x+2)\cdot(3-x)}$$

- $$f(x) = \log|x+2| - \log|3-x| = \log|x+2| - \log|x-3|$$
$$f'(x) = \frac{1}{x+2} - \frac{1}{x-3} = \frac{x-3-x-2}{(x+2)(x-3)} = -\frac{5}{(x+2)(x-3)}$$

⑥ $e^{-3x}(x^2+2x-1)$

$$(-3\cdot e^{-3x})\cdot(x^2+2x-1) + e^{-3x}\cdot(2x+2)$$

$$= e^{-3x}\cdot(-3x^2-4x+5)$$

⑦ $\boxed{\frac{1}{a}\cdot \arctan\left(\frac{x}{a}\right)}$ $\qquad a>0$

$$\frac{1}{a}\cdot\frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1}{a} = \frac{1}{a^2+x^2}$$

⑨ $\arctan \frac{1+x}{1-x}$

$$\frac{1}{1+\left(\frac{1+x}{1-x}\right)^2}\cdot\frac{1\cdot(1-x)-(-1)\cdot(1+x)}{(1-x)^2} = \frac{2}{(1-x)^2+(1+x)^2}$$

$$= \frac{1}{1+x^2} = D(\arctan(x))$$

<u>Oss</u>: se $f'\equiv 0$ su $[a,b]$ allora $f\equiv c$ su $[a,b]$.

Scoperto $D\left(\arctan\frac{1+x}{1-x}\right) = D(\arctan(x))$

$$\Rightarrow \boxed{\arctan \frac{1+x}{1-x} = \arctan(x) + \text{cost.}}$$

Infatti: pongo $\vartheta = \arctan(x)$

cioè $\tan(\vartheta) = x$

Scrivo $s = \sin(\vartheta)$ $\quad c = \cos(\vartheta)$

$$\frac{1+x}{1-x} = \frac{1+\frac{s}{c}}{1-\frac{s}{c}} = \frac{c+s}{c-s} = \frac{\frac{1}{\sqrt{2}}\cdot c + \frac{1}{\sqrt{2}}\cdot s}{\frac{1}{\sqrt{2}}\cdot c - \frac{1}{2}\cdot s}$$

$$= \frac{\sin(\frac{\pi}{4})\cdot c + \cos(\vartheta)\cdot s}{\cos(\frac{\pi}{4})\cdot c - \sin(\frac{\pi}{4})\cdot s} = \frac{\sin(\frac{\pi}{4}+\vartheta)}{\cos(\frac{\pi}{4}+\vartheta)}$$

$$= \operatorname{tg}\left(\frac{\pi}{4}+\vartheta\right)$$

$$\boxed{\arctan\left(\frac{1+x}{1-x}\right) = \frac{\pi}{4}+\vartheta = \arctan(x) + \frac{\pi}{4}}$$

(10) $e^{2x}\cdot\left(2\cdot\sin(3x) - 4\cdot\cos(3x)\right)$

$$2\cdot e^{2x}\cdot\left(2\sin(3x) - 4\cos(3x)\right) + e^{2x}\cdot\left(6\cos(3x) + 12\cdot\sin(3x)\right)$$

$$= e^{2x}\cdot\left(16\sin(3x) - 2\cos(3x)\right).$$

(12) $\cotg(x)$, $\tanh(x)$, $\cotanh(x)$

$$D(\cotg(x)) = D\left(\frac{\cos(x)}{\sin(x)}\right) = \frac{-\sin^2(x)-\cos^2(x)}{\sin^2(x)} = -\frac{1}{\sin^2(x)}$$

$$= -1-\cotg^2(x)$$

$$D(\tanh(x)) = D\left(\frac{\sinh(x)}{\cosh(x)}\right) = \frac{\cosh^2(x)-\sinh^2(x)}{\cosh^2(x)}$$

$$= \frac{1}{\cosh^2(x)} = 1-\tanh^2(x)$$

$$D(\cotanh(x)) = D\left(\frac{\cosh(x)}{\sinh(x)}\right) = \frac{\sinh^2(x)-\cosh^2(x)}{\sinh^2(x)}$$

$$= -\frac{1}{\sinh^2(x)} = 1-\cotanh^2(x)$$

(13) 2^{x^2+3x}

$$\log(2)\cdot 2^{x^2+3x}\cdot(2x+3)$$

$D(a^x)$ $a>0$

$$= D\left(\left(e^{\log(a)}\right)^x\right) = D\left(e^{x\cdot\log(a)}\right)$$

$$= e^{x\cdot\log(a)}\cdot\log(a) = a^x\cdot\log(a)$$

$$\boxed{D(a^x) = \log(a)\cdot a^x}$$

(15) $\log_2 |3x|$

$$D(\log_a(x)) = D\left(\frac{\log(x)}{\log(a)}\right) = \frac{1}{\log(a)\cdot x}$$

- $$\frac{1}{\log(2)\cdot 3x}\cdot 3 = \frac{1}{\log(2)\cdot x}$$

- $$\log_2|3x| = \underbrace{\log_2(3)}_{0} + \underbrace{\log_2|x|}_{\frac{1}{\log(2)\cdot x}}$$

(17) $x^{x\cdot\log(x)}$

$$D(x^x) = D\left(e^{\log(x^x)}\right) = D\left(e^{x\cdot\log(x)}\right) = e^{x\cdot\log(x)}\cdot\left(\log(x) + x\cdot\frac{1}{x}\right) = (1+\log(x))\cdot x^x$$

$$D\left(x^{x\cdot\log(x)}\right)$$

$$= D\left(e^{\log\left(x^{x\cdot\log(x)}\right)}\right)$$

$$= D\left(e^{x\cdot\log^2(x)}\right)$$

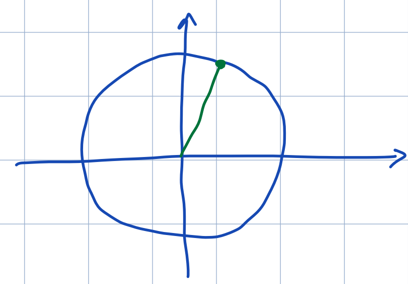
$$= x^{x\cdot\log(x)}\cdot\left(1\cdot\log^2(x) + x\cdot 2\log(x)\cdot\frac{1}{x}\right)$$

$$= x^{x\cdot\log(x)}\cdot\left(\log(x) + 2\right)$$

Trovare retta tangente al grafico di f in punto di ascissa x_0.

tangente : $y = f(x_0) + f'(x_0)\cdot(x - x_0)$

(20) $\sin(x)$ $\quad x_0 = \frac{\pi}{3}$



$f(x_0) = \sin\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$f'(x_0) = \cos\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ $\qquad y = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)$

(21) $f(x) = (x \cdot \log|x|)^3$ $\qquad x_0 = -1$

$f(x_0) = 0$

$f'(x) = 3\cdot(\underbrace{x\cdot\log|x|}_{0})^2 \cdot \left(\log|x| + x\cdot\frac{1}{x}\right)$

$f'(x_0) = 0$ $\qquad y = 0$

Trovare i punti di non derivabilità e dire di che tipo sono.

(32) $|x^2 + 3x - 4|$

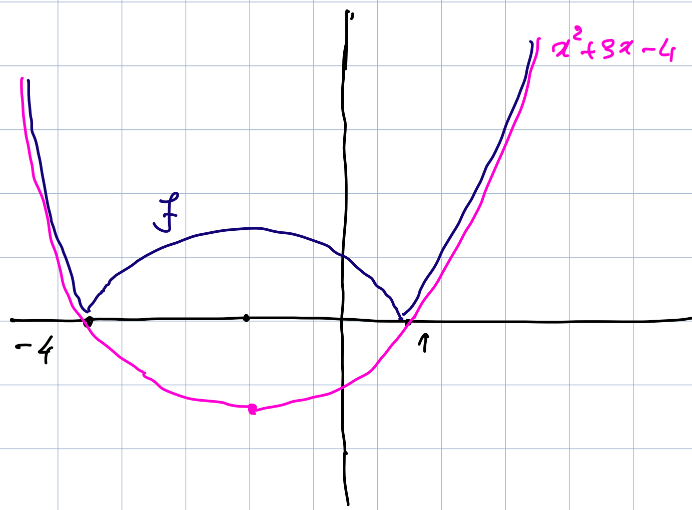
Oss: se f è derivabile su I allora $|f|$ è derivabile su I tranne al più negli zeri di f.

$f(x) = |(x+4)(x-1)|$

$\lim_{x\to 1^\pm} \frac{f(x) - f(1)}{x-1} =$

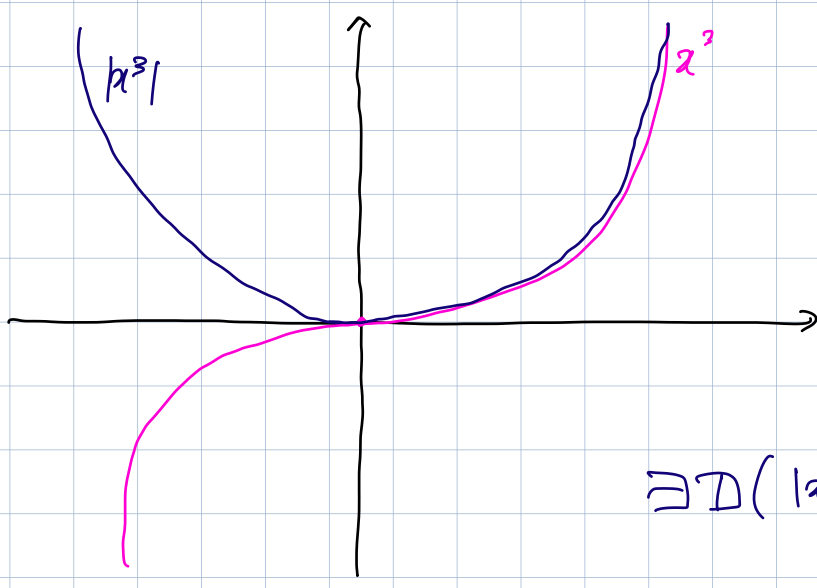
Es: $|x^2| = x^2$ der. su $\mathbb{R}$

$$= \lim_{x\to 1^\pm} \frac{|x+4|\cdot|x-1|}{x-1} = \pm 5 \implies \nexists\, f'(1)$$



$\nexists\, f'(-4)$

Es: $|x^3| = |x|^3$



$\exists\, D(|x^3|)$ in 0 e fa 0

(34) $\sqrt[3]{2x^2+x-1}$ Dove $2x^2+x-1 \neq 0$ $\exists f'(x)$

$$2x^2+x-1 = (2x-1)(x+1)$$

nulle in $-1, \frac{1}{2}$

$\exists f'(-1)$?

$$\lim_{x\to -1} \frac{\sqrt[3]{(2x-1)\cdot(x+1)}}{x+1} = \lim_{x\to -1} \frac{\sqrt[3]{2x-1}}{\sqrt[3]{(x+1)^2}} = \frac{-\sqrt[3]{3}}{0^+} = -\infty$$

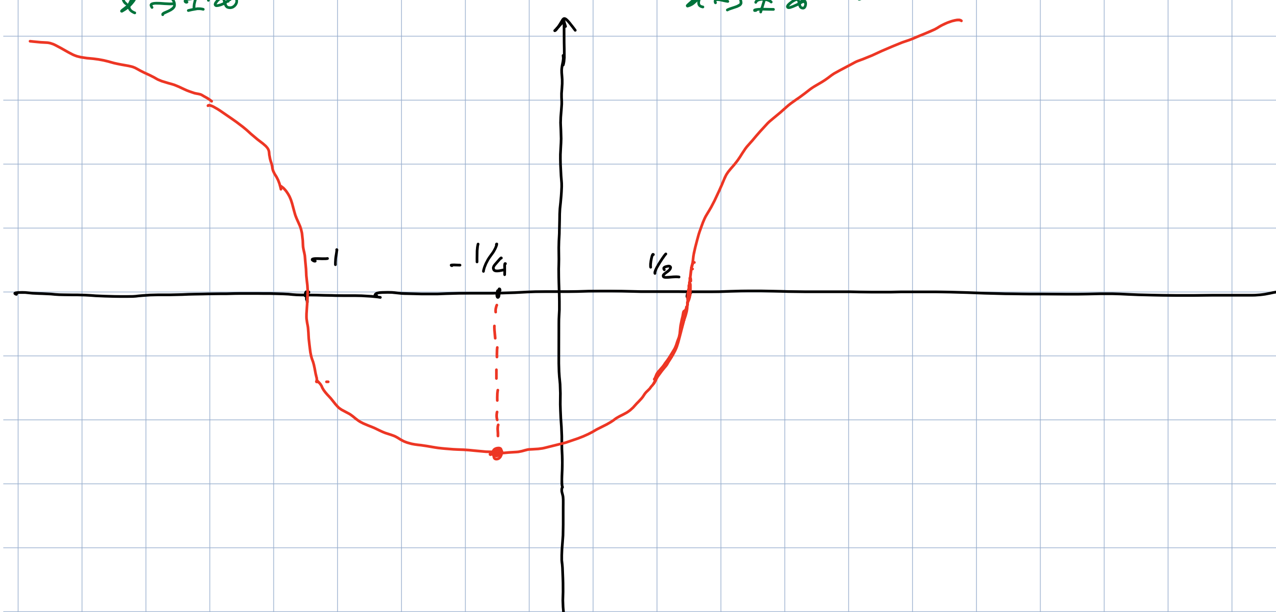
$$\lim_{x\to \frac{1}{2}} = \frac{\sqrt[3]{3/2}}{0^+} = +\infty$$

Altrove: $f'(x) = \frac{1}{3}(2x^2+x-1)^{-2/3} \cdot (4x+1)$

cambia segno per $x = -1/4$

da $-$ a $+$ $\Rightarrow$ ha min. loc.

$$\lim_{x\to\pm\infty} f(x) = +\infty \qquad \lim_{x\to\pm\infty} \frac{f(x)}{x} = 0$$



$$f'(x) = \frac{1}{3}\left(2x^2+x-1\right)^{-2/3} \cdot (4x+1)$$

$$f''(x) = -\frac{2}{9}\left(2x^2+x-1\right)^{-5/3} \cdot (4x+1)^2 + \frac{1}{3}\left(2x^2+x-1\right)^{-2/3} \cdot 4$$

$$= -\frac{2}{9}\left(2x^2+x-1\right)^{-5/3} \cdot \left(16x^2+8x+1-6\cdot\left(2x^2+x-1\right)\right)$$

$$= -\frac{2}{9}\left(2x^2+x-1\right)^{-5/3} \cdot \left(\underbrace{4x^2+2x+7}\right)$$

$$\Delta = 1-4\cdot4\cdot7 < 0$$

sempre pos.