Ist. Mat. I - CIA

16/11/23

$$\sum_{n=0}^{\infty} a_n = \lim_{k\to\infty} \underset{\displaystyle\sum_{n=0}^{k} a_n}{\underset{\parallel}{S_k}}$$

Es: $\sum_{n=0}^{\infty} \alpha^n$ $\qquad S_k = \sum_{n=0}^{k} \alpha^k = \dfrac{1-\alpha^{n+1}}{1-\alpha}$

$$\sum_{n=0}^{\infty} \alpha_n = \begin{cases} \dfrac{1}{1-\alpha} & |\alpha|<1 \\ \nexists & |\alpha|\geq 1 \end{cases}$$

Oss: se $\sum_{n=0}^{\infty} \alpha_n$ $\overbrace{\text{converge}}^{(a\ S)}$ allora $\alpha_n \to 0$.

Ragione $\quad \alpha_n = \underbrace{\underset{\downarrow}{S_n} - \underset{\downarrow}{S_{n-1}}}_{S \quad\quad S}$

$\downarrow$

0.

Q: $\sum_{n=1}^{\infty} \frac{1}{n}$ Converge?

$S_k = \dots$

$k=10$	$2.92\dots$
$k=100$	$5.18\dots$
$k=1000$	$7.48\dots$
$k=10000$	$9.78\dots$
$\vdots$	

Fatto: fa $+\infty$. (Oss: S_k cresce di circa ~ 2.3 ogni fattore 10 per k)

Oss: se $a_n = b_n - b_{n+1}$ con $b_n \to 0$ allora
$\sum_{n=0}^{\infty} a_n = b_0$.

Infatti: $S_k = a_0 + a_1 + \dots + a_k$

$= (b_0 - \cancel{b_1}) + (\cancel{b_1} - \cancel{b_2}) + (\cancel{b_2} - b_3) + \dots + (b_k - b_{k+1})$

$= b_0 - \underbrace{b_{k+1}}_{\downarrow\, 0}$

Es: $b_n = \frac{1}{n}$ $\quad a_n = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Data $\sum_{n=0}^{+\infty} a_n$ poniamo

$S_k = \sum_{n=0}^{k} a_n$ somma parziale k-esima

$R_k = \sum_{n=k+1}^{+\infty} a_n$ resto k-esimo

Oss: se $\sum a_n$ converge allora R_k converge $\forall k$ e tende a 0 per $k \to +\infty$.

Infatti la somma parziale m-esima di R_k è

$$\sum_{n=k+1}^{m} a_n = S_m - S_k \qquad (k \text{ fisso}, m \to +\infty)$$

$$\underbrace{\underset{S}{\downarrow} \quad \underset{\text{fisso}}{|}}_{\downarrow\; S - S_k \;\longrightarrow\; 0}$$

Oss: se $a_n \geq 0$ $\forall n$ allora $\sum_{n=0}^{+\infty} a_n$ o converge o diverge a $+\infty$ poiché S_k è crescente

Criteri:

- $0 \leq a_n \leq b_n$; se $\sum b_n < +\infty$ allora $\sum a_n < +\infty$
  se $\sum a_n = +\infty$ allora $\sum b_n = +\infty$

Conseguenze:

- $\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\qquad \frac{1}{n^2} \leq \frac{1}{n(n+1)}$

  ma $\sum \frac{1}{n(n+1)} < +\infty$

  $\Rightarrow \sum \frac{1}{n^2} < +\infty$

- $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ per $\alpha \geq 2$ $\qquad \frac{1}{n^\alpha} \leq \frac{1}{n^2}$

  $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^\alpha} < +\infty$

- assumendo $\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$ ho se $\alpha < 1$

  $\frac{1}{n^\alpha} \geq \frac{1}{n} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^\alpha} = +\infty$

Fatto (vedremo): $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^\alpha} = \begin{cases} < +\infty & \text{se } \alpha > 1 \\ +\infty & \text{se } \alpha \leq 1 \end{cases}$

- (rapporto): $a_n > 0$; se $\dfrac{a_{n+1}}{a_n} \to L$ allora

$$\sum_{n=0}^{\infty} a_n = \begin{cases} < +\infty & \text{se } L < 1 \\ +\infty & \text{se } L > 1 \\ ??? & \text{se } L = 1. \end{cases}$$

Ragione: $L < 1$ scelgo ε t.c. $L + \varepsilon < 1$.
$\exists N$ t.c. $\dfrac{a_{n+1}}{a_n} < L + \varepsilon \quad \forall n \geq N$.

$$a_{N+1} < a_N \cdot (L+\varepsilon)$$
$$a_{N+2} < a_{N+1} \cdot (L+\varepsilon) < a_N \cdot (L+\varepsilon)^2$$
$$a_{N+3} < a_{N+2} \cdot (L+\varepsilon) < a_N \cdot (L+\varepsilon)^3$$
$$\vdots$$
$$a_{N+k} < a_N \cdot (L+\varepsilon)^k$$

$$\Longrightarrow \sum_{n=N}^{N+k} a_n < a_N \cdot \underbrace{\sum_{m=0}^{k} (L+\varepsilon)^m}$$

convergente poiché $L+\varepsilon < 1$

$\Longrightarrow$ convergente

Se $L > 1$ scelgo ε con $L - \varepsilon > 1$; $\exists N$ t.c.

$$\frac{a_{n+1}}{a_n} > L - \varepsilon \quad \forall n \geq N$$

$$\Longrightarrow \quad a_{N+k} > a_N \cdot (L-\varepsilon)^k$$

$$\sum_{n=N}^{N+k} a_n > a_N \cdot \underbrace{\sum_{n=0}^{k} (L-\varepsilon)^k}_{\downarrow \; +\infty \text{ poiché } L-\varepsilon>1}$$

$$\downarrow$$

$$+\infty$$

- se $a_n > 0 \quad \forall n$; se $\lim_{n\to\infty} \sqrt[n]{a_n} = L$ allora

se $L<1$ $\qquad \sum a_n < +\infty$

se $L>1$ $\qquad \sum a_n = +\infty$

se $L=1$ $\qquad$???

Se $L<1$ scelgo ε con $L+\varepsilon<1$ e $\exists N$ t.c.

$$\sqrt[n]{a_n} < L+\varepsilon \qquad \forall n \geq N$$

$$\Rightarrow a_n < (L+\varepsilon)^n \qquad \forall n \geq N$$

$$\sum (L+\varepsilon)^n < +\infty \quad \text{poiché } L+\varepsilon<1$$

$$\Rightarrow \sum a_n < +\infty$$

$L>1$ analogo.

<u>Es</u>: $\quad \sum \left(\frac{1}{2}+\frac{1}{n}\right)^n$

$$\sqrt[n]{\left(\frac{1}{2}+\frac{1}{n}\right)^n} = \frac{1}{2}+\frac{1}{n} \longrightarrow \frac{1}{2}$$

$\Rightarrow$ converge.

Def: dico che $\sum a_n$ è ass. conv. se $\sum |a_n|$ è convergente.

Prop: ass. conv. $\Rightarrow$ conv.

Dimo: chiamo $S_n^+ = \sum_{\substack{k=0 \\ a_k \geq 0}}^{n} a_k$; $S_n^- = \sum_{\substack{k=0 \\ a_k < 0}}^{n} (-a_k)$

$$\sum_{k=0}^{n} |a_k| = S_n^+ + S_n^-$$

Oss: $S_n^+ \nearrow$, $S_n^- \nearrow$; se $\sum |a_k| < +\infty$ e sono ≥ 0

$\Rightarrow$ sono limitate $\Rightarrow S_n^\pm$ convergono.

Ma $\sum_{k=0}^{n} a_k = S_n^+ - S_n^- \Rightarrow$ converge. ∎

Fatto: falso il viceversa.

Vedremo tra un attimo: $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$ converge, ma

$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$ (da dimostrare)

Esempio: $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ (serie di Taylor di e^x)

Dimostro che converge: fa 1 per $x=0$;

Se $x>0$; $a_n = \dfrac{x^n}{n!}$ $\qquad \dfrac{a_{n+1}}{a_n} = \dfrac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} = \dfrac{x}{n+1} \to 0$

$\Rightarrow$ converge per crit. rapp.

$x<0$ $\qquad |a_n| = \dfrac{|x|^n}{n!} \Rightarrow \sum |a_n| < +\infty$

$\Rightarrow \sum a_n$ convergente.

Fatto: $\displaystyle\sum_{n=0}^{+\infty} \frac{x^n}{n!} = e^x$.

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

pari; continua; $\displaystyle\lim_{x\to\pm\infty} f(x) = 1^-$

$f'(x) = e^{-\frac{1}{x^2}} \cdot \dfrac{2}{x^3}$ ha lo stesso segno di x

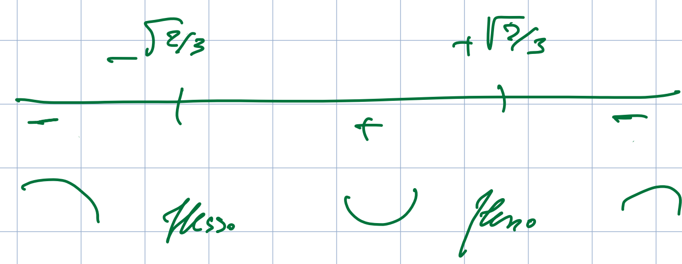
$$\begin{array}{ccc} & 0 & \\ \searrow & \text{min} & \nearrow \end{array}$$

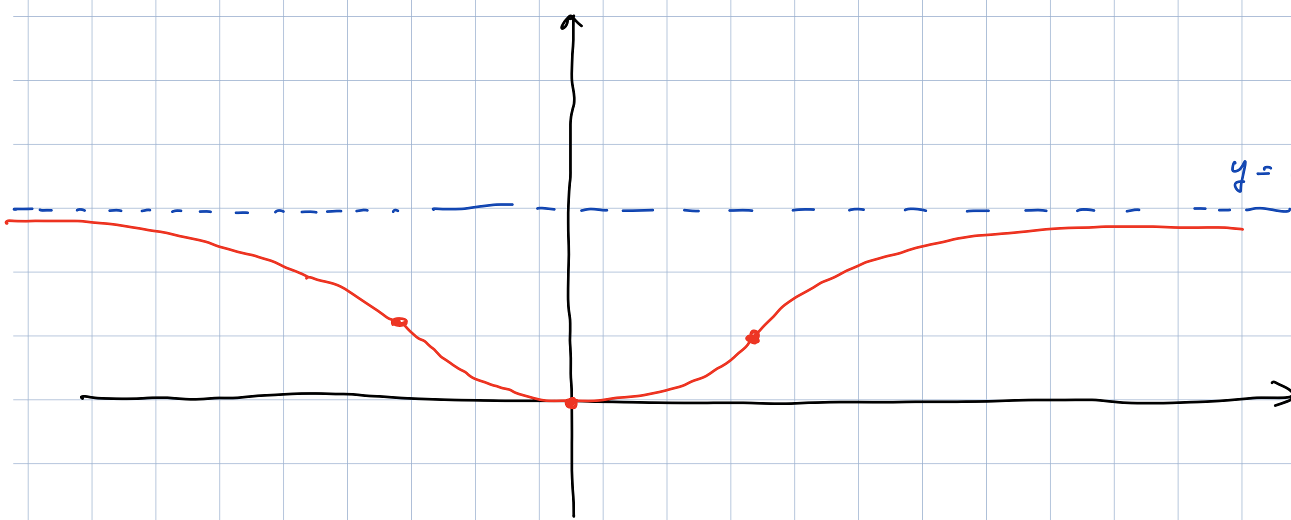
$$f''(x) = e^{-\frac{1}{x^2}} \cdot \left(\frac{2}{x^3}\right)^2 + e^{-\frac{1}{x^2}} \cdot \left(-\frac{6}{x^4}\right)$$

$$= \frac{2}{x^4}\, e^{-\frac{1}{x^2}} \left(\frac{2}{x^2} - 3 \right)$$

cambia segno

$x = \pm \sqrt{2/3}$





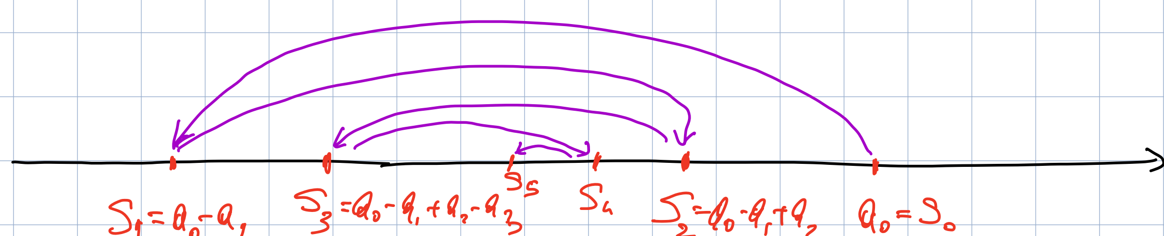
— o —

Teo (crit. di Leibniz): $(a_n)_{n=0}^{+\infty}$ t.c.

$a_n > 0$, $a_n \searrow$, $a_n \to 0$

$$\Longrightarrow \sum_{n=0}^{\infty} (-1)^n \cdot a_n \text{ converge.}$$

(serie a termini di segno alterno)



Dim: $b_m = S_{2m}$ $\quad c_m = S_{2m+1}$

$$b_{m+1} - b_m = S_{2m+2} - S_{2m}$$
$$= a_{2m+2} - a_{2m+1} < 0 \quad \text{poiché } a_m \searrow 0$$

$\Rightarrow b_m \searrow$

$$c_{m+1} - c_m = S_{2m+3} - S_{2m+1}$$
$$= -a_{2m+3} + a_{2m+2} > 0 \quad \text{poiché } a_m \searrow 0$$

$\Rightarrow c_m \nearrow$

$$b_m - c_m = S_{2m} - S_{2m+1} = a_{2m+1} > 0$$

$\Rightarrow c_m < b_m$,

$\Rightarrow c_m \searrow L^+ \quad b_m \nearrow L^-$

$$b_m - c_m = a_{2m+1} \longrightarrow 0$$

$\Rightarrow L^+ = L^-$. $\quad \blacksquare$

Es: $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\log(n)}$ converge

ma (fatto): $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\log(n)} = +\infty$, Foglio 5 Es 1 : si applicano Weierstrass / zeri

(d) $f : [0,+\infty) \to \mathbb{R}$ $\quad f(x) = \frac{1}{1+x}$

$[0,+\infty)$ no chiuso no limitato $\Rightarrow$ NO.

Non ci sono zeri: $f(x) > 0 \ \forall x$.

f è decrescente: ha max 1 in $x=0$

non ha min: $\lim_{x\to+\infty} f(x) = 0^+$

(e) $f : [0,1) \to \mathbb{R}$ $\quad f(x) = \frac{1}{\log(2-x)}$

$[0,1)$ no chiuso

$$\lim_{x\to 1^-} f(x) = \lim_{t\to 1^+} \frac{1}{\log(t)} = \frac{1}{0^+} = +\infty$$

Sempre positiva $\Rightarrow$ non ha zeri.

crescente:

$$f'(x) = -\frac{1}{(\log(2-x))^2} \cdot \left(\frac{1}{2-x}\right) \cdot (-1) > 0$$

crescente.

$\boxed{2}$ Trovare $\mathrm{Im}(f)$

(a) $f : [0,4] \to \mathbb{R}$ $\quad f(x) = x + \sqrt{x} - 1$

f è crescente : cresc + cresc + cost

$$\mathrm{Im}(f) = [f(0), f(4)] = [-1, 5]$$

(b) $f : [-1,1] \to \mathbb{R}$ $\qquad f(x) = \arctan(x) - \sqrt{1-x}$

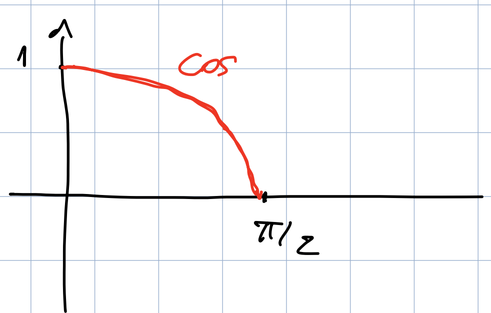
f é crescente: $\arctan \nearrow$

$1-x \searrow$; $\sqrt{1-x} \searrow$; $-\sqrt{1-x} \nearrow$

$\Rightarrow f = \text{cresc} + \text{cresc}$

$$\mathrm{Im}(f) = [f(-1), f(1)] = \left[-\frac{\pi}{4} - \sqrt{2}, \frac{\pi}{4}\right]$$

(c) $f : [0, \frac{\pi}{2}] \to \mathbb{R}$ $\qquad f(x) = \cos(x) - x^3$



f decr: decr − cresc

decr + decr

$$\mathrm{Im}(f) = \left[f\left(\frac{\pi}{2}\right), f(0)\right] = \left[-\frac{\pi^3}{8}, 1\right]$$

(d) $f : [0,2] \to \mathbb{R}$ $\qquad f(x) = \sin(x^3)$

$f = \sin \circ g$ $\qquad g(x) = x^3$

$$\mathrm{Im}(g) = [0, 8]$$

$$\mathrm{Im}(f) = \mathrm{Im}\left(\mathrm{sen}|_{[0,8]}\right) = [-1, 1].$$ porque $[0,8] \supset [0, 2\pi]$

$\boxed{3}$ Data $f: I \to \mathbb{R}$ trovare $J = \mathrm{Im}(f)$ e dire se $g: I \to J$ $(g(x) = f(x)\ \forall x)$ è invertibile con inversa continua

(a) $f: [2,4] \to \mathbb{R}$ $\quad f(x) = x - 2\sqrt{x}$

f è continua; $\quad f'(x) = 1 - 2 \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{x}} = 1 - \frac{1}{\sqrt{x}} > 0$ su $[2,4]$

$J = [f(2), f(4)] = [2 - 2\sqrt{2}, 0]$

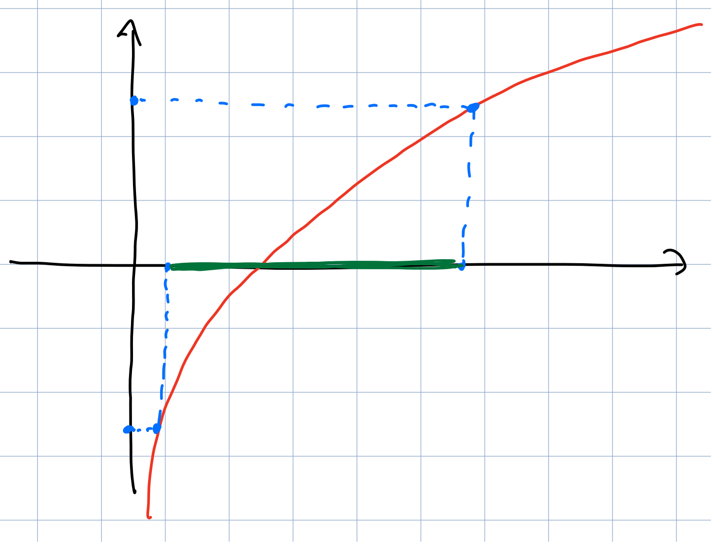
f^{-1} esiste continua.

(b) $f: (0, +\infty) \to \mathbb{R}$ $\quad f(x) = e^x + \log(x)$

f continua ; $f = \text{cresc} + \text{cresc} = \text{crescente}$

$\lim_{x \to 0^+} f(x) = 1 + (-\infty) = -\infty$

$\lim_{x \to +\infty} f(x) = +\infty + \infty = +\infty.$



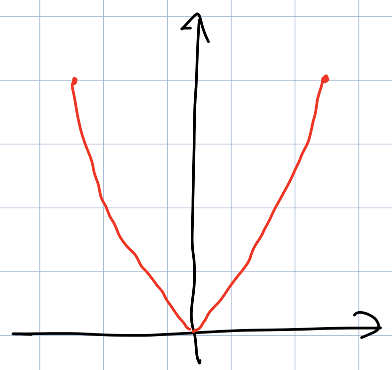
$\mathrm{Im}(f) = \mathbb{R}$

f è invertibile inversa continua

(c) $f: [-1,1] \to \mathbb{R}$ $\qquad f(x) = x^2 + |x|$

f è pari : crescente su $[0,1]$, decr. su $[-1,0]$

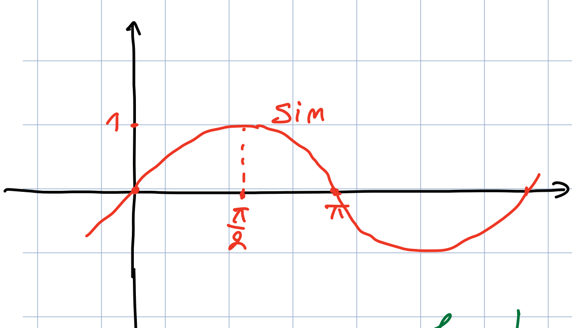
$\mathrm{Im}(f) = [f(0), f(1)] = [0,2]$

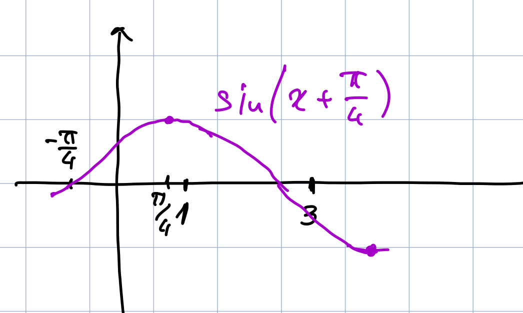


(d) $f: [1,3] \to \mathbb{R}$ $\qquad f(x) = \sin(x) + \cos(x)$.

$$f(x) = \sqrt{2} \cdot \Big(\underbrace{\frac{1}{\sqrt{2}}}_{=\cos\left(\frac{\pi}{4}\right)} \cdot \sin(x) + \underbrace{\frac{1}{2}}_{=\sin\left(\frac{\pi}{4}\right)} \cdot \cos(x) \Big)$$

$$= \sqrt{2} \cdot \sin\left(x + \frac{\pi}{4}\right)$$





f decrescente $\Rightarrow$ invertibile con inv. cont.