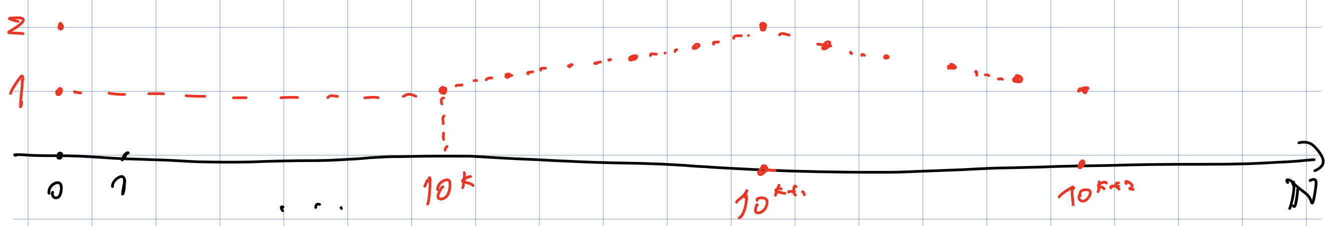
Ist. Mat. I - CIA
26/10/23

$\lim \frac{a_{n+1}}{a_n} = 1$ non si può concludere

$\lim (a_n)$ può anche non esistere:

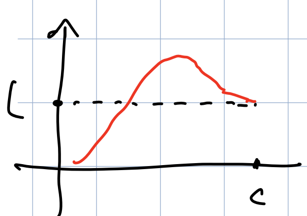


— o —

$I \subset \mathbb{R}$, $c \in \overline{I}$, $L \in \overline{\mathbb{R}}$, $f : I \to \mathbb{R}$

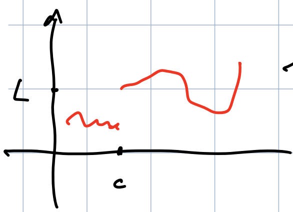
$$\lim_{x \to c} f(x) = L$$

Varianti:



$\lim_{x \to c} f(x) = L^+$ se $\lim_{x \to c} f(x) = L$ e $f(x) > L$ per x vicino a c

$\lim_{x \to c} f(x) = L^-$ . . . $f(x) < L$



$$\lim_{x\to c^+} f(x) = L$$

(caso finito)
$\forall \varepsilon>0\ \exists \delta>0$ t.c. $|f(x)-L|<\varepsilon$ per $|x-c|<\delta$, $x>c$

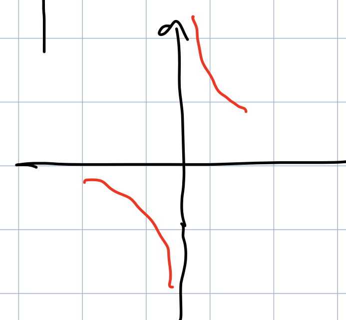
$$\lim_{x\to c^-} f(x) = L$$

. . . $x<c$

$$\lim_{x\to c^\pm} f(x) = L^\pm$$

Es: $\lim_{x\to+\infty} \frac{1}{x} = 0^+$

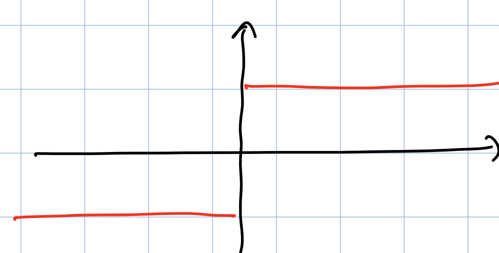
$\lim_{x\to 0} \frac{1}{x}$ non esiste



$\lim_{x\to 0^\pm} \frac{1}{x} = \pm\infty$

(Convenzione: se scrivo solo $f(x)$ senza dire dov'è I dove f è definita intendo I è il più grande possibile.)

$\lim_{x\to 0} \frac{|x|}{x}$ non esiste



$\lim_{x\to 0^\pm} \frac{|x|}{x} = \pm 1$

Teo: $\lim_{x \to 0} \sin(x) = 0$.

ATT: Non è perché $\sin(0) = 0$.

Es: $f(x) = \begin{cases} x & \text{per } x \neq 0 \\ 2 & \text{per } x = 0 \end{cases}$

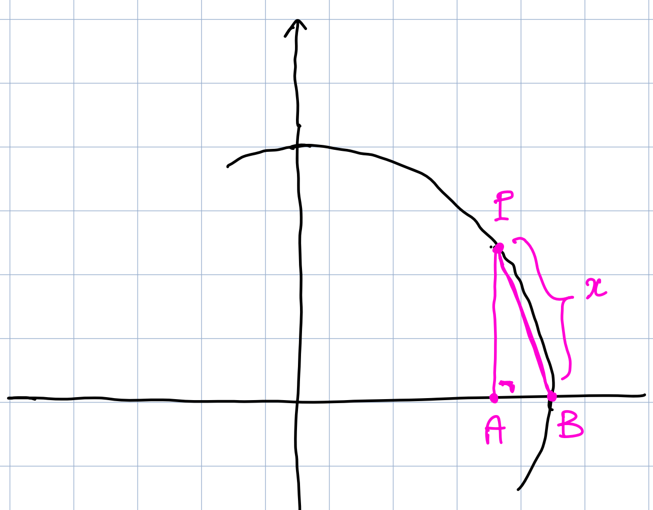
$$\lim_{x \to 0} f(x) = 0 \neq f(0).$$

Dimo: devo vedere che se x è vicino a 0 allora $\sin(x)$ è vicino a 0.

Dimostro che per $x \neq 0$ (piccolo) ho

$$0 < |\sin(x)| < |x|.$$

Basta vederla per $x > 0$:



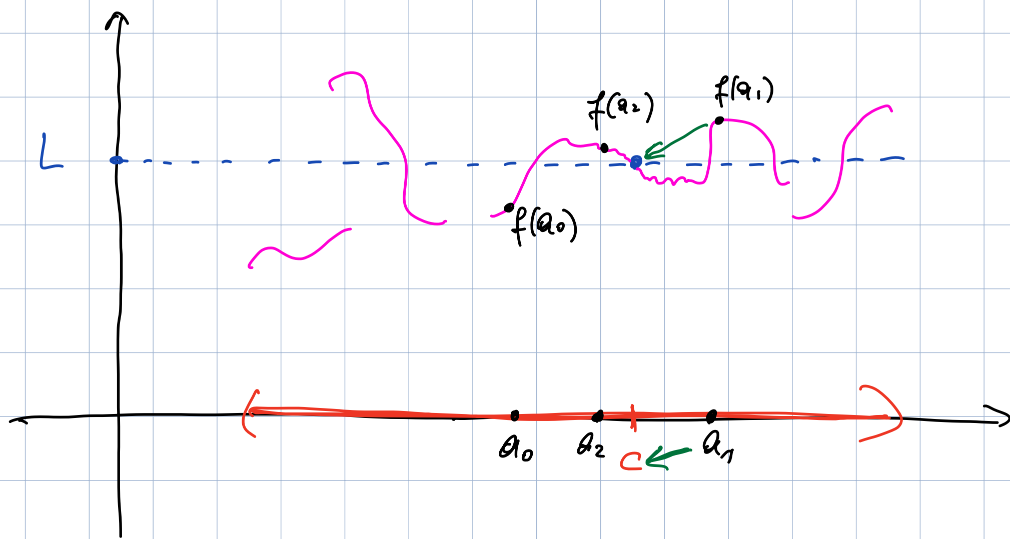
$\overline{AP} = \sin(x)$
$x = \overset{\frown}{BP}$

$$\underset{\sin(x)}{\overline{AP}} = \sqrt{\overline{BP}^2 - \overline{AB}^2} < \overline{BP} < \underset{x}{\overset{\frown}{BP}}.$$

$\blacksquare$

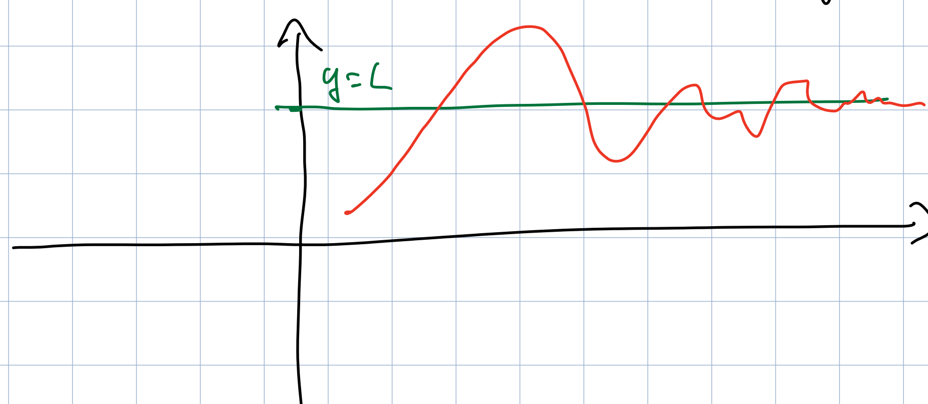
Esercizio: $f: I \to \mathbb{R}$

$$\lim_{x \to c} f(x) = L \iff \forall (a_n) \text{ t.c. } a_n \in I,\ a_n \neq c,\ \lim (a_n) = c \text{ si ha } \lim f(a_n) = L$$

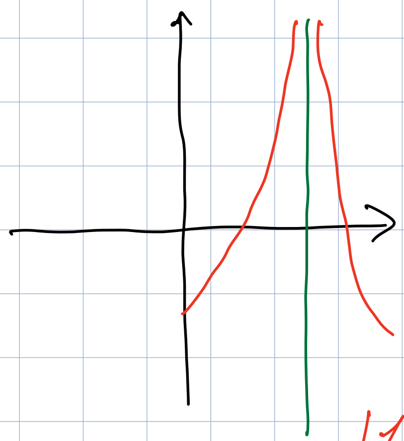


Asintoti del grafico di una funzione

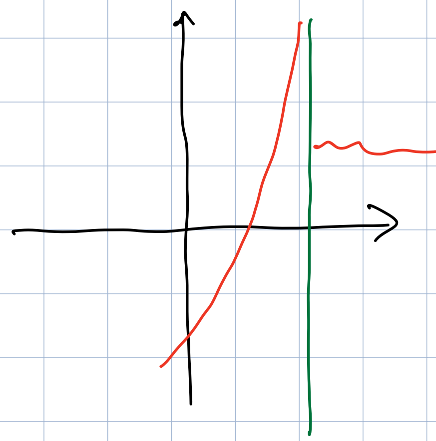
$I = (\dots, +\infty)$ $\quad \lim_{x \to +\infty} f(x) = L \in \mathbb{R}$ $\quad$ asintoto orizzontale destro $y = L$



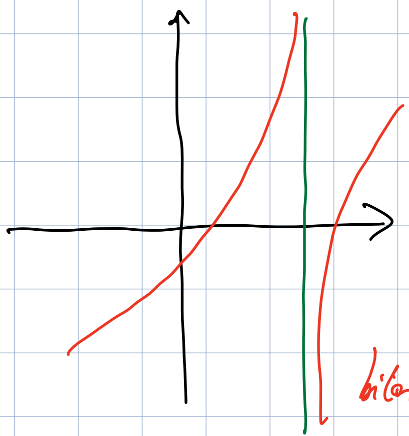
$I \ni c \in \mathbb{R}$ $\quad \lim_{x \to c^{\pm}} f(x) = \pm\infty$ $\quad$ asintoto verticale dx/sx $x = c$



bilat.



sx



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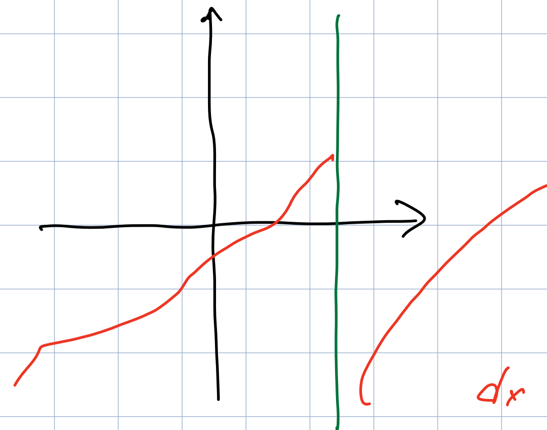
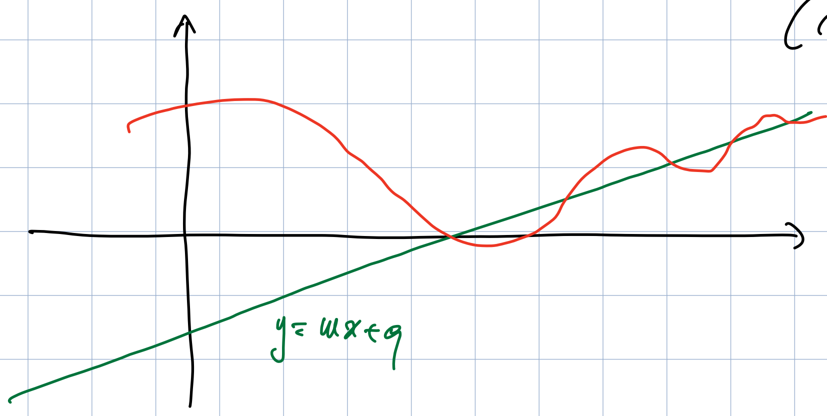


dx

Asintoto obliquo: $I = (\ldots, +\infty)$

se $\lim_{x \to +\infty} \Big(f(x) - (mx+q) \Big) = 0$ dico che f ha asintoto obliquo $y = mx+q$ $(m \neq 0)$



Oss: se $\lim_{x\to\infty}\left(f(x)-(mx+q)\right)=0$ ho

$$m=\lim_{x\to\infty}\frac{f(x)}{x} \qquad q=\lim_{x\to\infty}\left(f(x)-m\cdot x\right)$$

$$0=\lim_{x\to\infty}\frac{f(x)-(mx+q)}{x}=\lim_{x\to\infty}\left(\frac{f(x)}{x}-\underbrace{m}_{\text{costante}}+\underbrace{\frac{q}{x}}_{\downarrow\, 0}\right)=\left(\lim_{x\to\infty}\frac{f(x)}{x}\right)-m$$

$q=\dots$

Altri limiti notevoli:

- $\lim_{x\to 0}\frac{\sin(x)}{x}=1$ (argomento geometrico simile al precedente)

(dimo domani)

- $\lim_{x\to 0}\cos(x)=1$ (dimo domani)

- $\lim_{x\to 0}\frac{1-\cos(x)}{x^2}=\frac{1}{2}$

Infatti

$$\frac{1-\cos(x)}{x^2}=\frac{1-\cos(x)}{x^2}\cdot\frac{1+\cos(x)}{1+\cos(x)}$$

$$=\frac{1-\cos^2(x)}{x^2}\cdot\frac{1}{1+\cos(x)}$$

$$= \frac{\sin^2(x)}{x^2} \cdot \frac{1}{1+\cos(x)}$$

$$= \underbrace{\left(\frac{\sin(x)}{x}\right)^2}_{\substack{\downarrow \\ 1 \\ 1^2 \\ \| \\ 1}} \cdot \frac{1}{1+\cos(x)}$$

$$\frac{1}{1+\cos(x)} \to \frac{1}{1+1} = \frac{1}{2}.$$

Ricordo: $\quad e = \sum_{n=0}^{\infty} \frac{1}{n!} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n.$

- $\displaystyle\lim_{x\to\pm\infty} \left(1+\frac{1}{x}\right)^x = e$

Limite della composizione:

supponiamo di avere funzioni f e g per cui abbia senso fare la composizione $f \circ g$

(cioè ho $f(y)$ e $g(x)$ ha senso calcolare $f(g(x))$ cioè porre $y = g(x)$ dentro f)

Se $\boxed{\lim_{x\to c} g(x) = d}$ $\quad \boxed{\lim_{y\to d} f(y) = L}$

allora $\lim_{x \to c} f(g(x)) = L$

$$f(\underbrace{\underbrace{g(x)}_{\downarrow \, d}}_{L}) \quad \text{per } x \to c$$

Sottigliezza: bisogna chiedere che $g(x) \neq d$ per x vicino a c

Altri limiti notevoli derivati da

$$\lim_{x \to \pm\infty} \left(1 + \frac{1}{x}\right)^x = e$$

- $\lim_{x \to 0^\pm} \frac{\log(1+x)}{x} = 1$

Uso il limite della composizione sostituendo $x = \frac{1}{y}$ ovvero $y = \frac{1}{x}$ dunque per $x \to 0^\pm$ ho $y \to \pm\infty$

$$\frac{1}{x} \cdot \log(1+x) = y \cdot \log\left(1 + \frac{1}{y}\right) = \underbrace{\log\Big(\underbrace{\left(1+\frac{1}{y}\right)^y}_{\downarrow \, e}\Big)}_{\log(e) = 1}$$

(usato $\lim_{t \to z} (\log(t)) = \log(z)$)

Ese foglio 2

$\boxed{9}$ (b) $$\left(9^{\sqrt{5}}\right)^{2+\sqrt{5}} \cdot \left(\frac{1}{3}\right)^{(2+\sqrt{5})^2}$$

$$= 3^{2\cdot\sqrt{5}\cdot(2+\sqrt{5})} \cdot 3^{-(2+\sqrt{5})^2}$$

$$= 3^{\cancel{4\sqrt{5}}+10-4-\cancel{4\sqrt{5}}-5} = 3$$

$\boxed{10}$ (c) $$\left(\log_3(2)-\log_2(3)\right)\cdot\frac{\log_2(27)}{\log_3(\sqrt{3})+\log_4(3)}$$

$$=\left(\frac{1}{\log_2(3)}-\log_2(3)\right)\cdot\frac{3\cdot\log_2(3)}{\frac{1}{2}+\frac{1}{2}\log_2(3)}$$

$$=6\cdot\frac{1-\left(\log_2(3)\right)^2}{\cancel{\log_2(3)}}\cdot\frac{\cancel{\log_2(3)}}{1+\log_2(3)}$$

$$=6\left(1-\log_2(3)\right).$$

$\boxed{11}$ calcolare z, $\Re(z)$, $\Im(z)$

(b) $$z=(8-3i)(5+4i)$$
$$=40+32i-15i+12=52+17i$$
$\Re=52$
$\Im=17$

(d) $$z=(7-3i)^{-1}=\frac{7+3i}{49+9}=\underbrace{\frac{7}{58}}_{\Re}+\underbrace{\frac{3}{58}}_{\Im}\cdot i$$

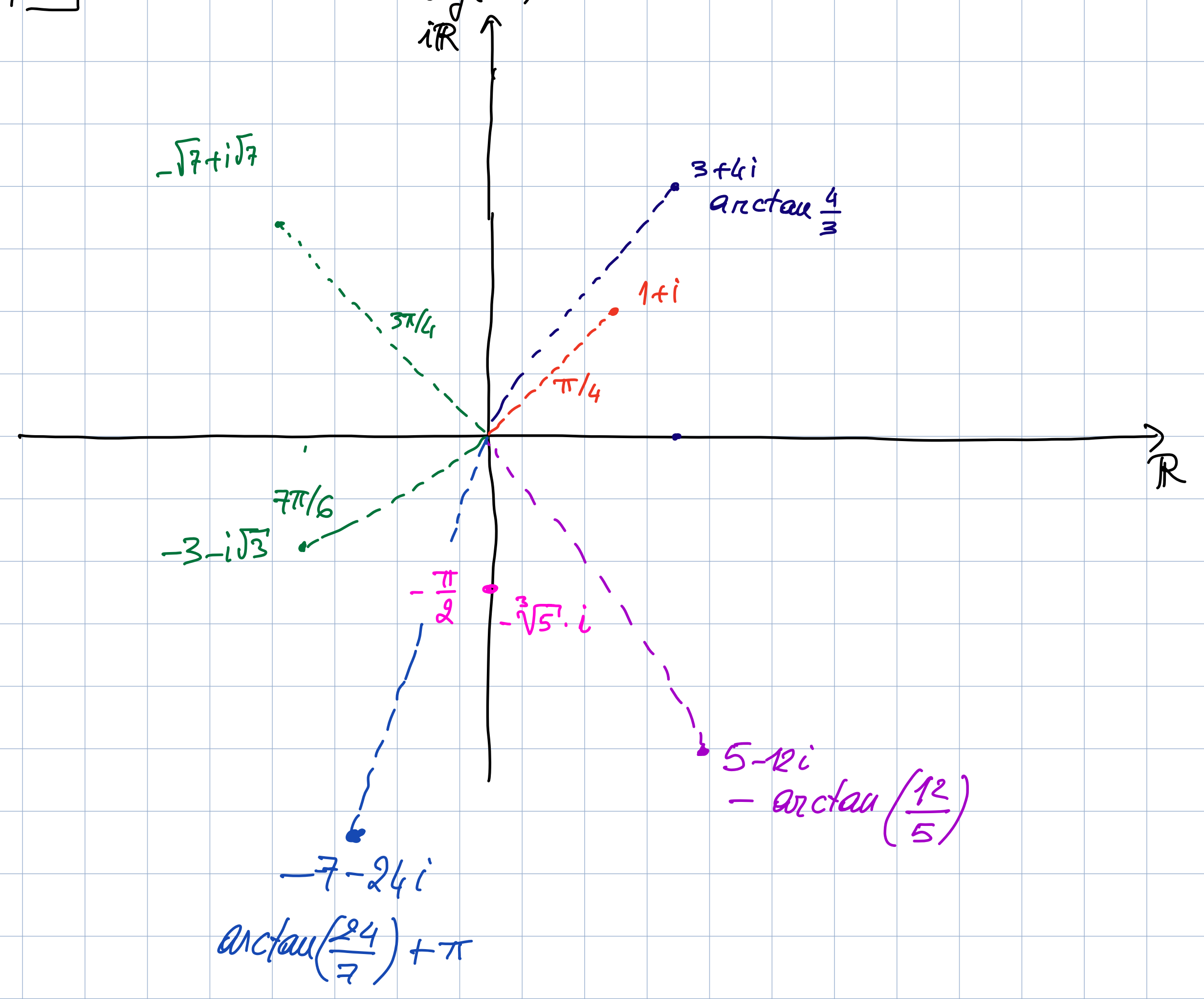
12 Calcolare z, $\bar{z}$, $|z|$

(b) $$z = \frac{\overline{2-3i} + 4\,|2-i|^2\, i}{7i + (1+5i)\cdot\overline{(4+i)}}$$

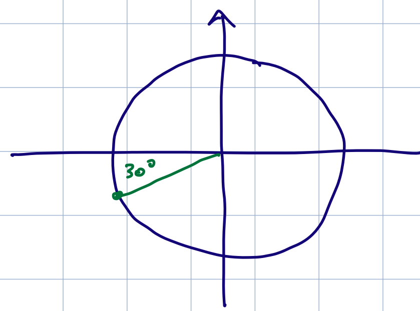
$$= \frac{2+3i+20i}{7i+(1+5i)(4-i)} = \frac{2+23i}{9+26i}$$

$$= \frac{(2+23i)\cdot(9-26i)}{9^2+26^2} = \dots$$

13 Trovare $\arg(z)$



$\boxed{14}$ calcolare e^z



(c) $z = \log(6) - i\frac{5}{6}\pi$

$$e^z = e^{\log(6)} \cdot e^{-i\frac{5}{6}\pi}$$
$$= 6 \cdot \left(\cos\left(-\frac{5}{6}\pi\right) + i \sin\left(-\frac{5}{6}\pi\right)\right)$$
$$= 6 \cdot \left(-\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right) = \dots$$

Foglio 3.

(2) Trovare radici reali con molt.

(a) $12x^3 - 32x^2 + 23x - 5$

So che se c'è una radice in $\mathbb{Q}$ è $\pm \dfrac{1/5}{1/2/3/4/6/12}$

$+1?$ $\quad 12 - 32 + 23 - 5 \neq 0$ $\quad$ No

$-1?$ $\quad -12 - 32 - 23 - 5 \neq 0$ $\quad$ No

$1/2?$ $\quad \frac{12}{8} - \frac{32}{4} + \frac{23}{2} - 5 = \frac{3}{2} - 8 + \frac{23}{2} - 5 = \frac{26}{2} - 13 = 0$ Sì

$$(12x^3 - 32x^2 + 23x - 5) = (2x-1)\cdot(6x^2 - 13x + 5)$$

$$
\begin{array}{c|cccc|c}
 & 12 & -32 & 23 & & -5 \\
1/2 & & 6 & -13 & & 5 \\
\hline
 & 12 & -26 & 10 & & / \\
\end{array}
$$

$$(\dots) = \left(x-\frac{1}{2}\right)\left(12x^2-26x+5\right)$$

$$x_1 = \frac{1}{2} \qquad x_{2,3} = \frac{13 \pm \sqrt{169-120}}{12} = \frac{13 \pm 7}{12} \begin{cases} 5/3 \\ 1/2 \end{cases}$$

$$(2x-1)^2(3x-5)$$

radice 1/2 con molt 2, radice 5/3 con molt. 1.

$\boxed{3}$ (a) $z^2 + iz + 2 = 0$

$$\Delta = i^2 - 4\cdot 1\cdot 2 = -1-8 = -9 \qquad \sqrt{\Delta} = \pm 3i$$

$$z_{1,2} = \frac{-i \pm 3i}{2} \begin{cases} i \\ -2i \end{cases}$$

Oss: $(z-z_1)\cdot(z-z_2) = z^2 - (z_1+z_2)\cdot z + z_1\cdot z_2$

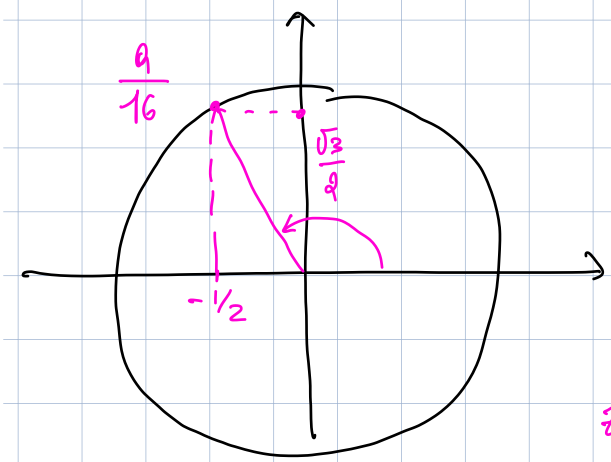
$= z^2 + bz + c$

$b = -(z_1+z_2) \qquad c = z_1\cdot z_2$

(e) $z^4 = -8+8\sqrt{3}\cdot i$ (le radici quarte di $a=-8+8\sqrt{3}\cdot i$)

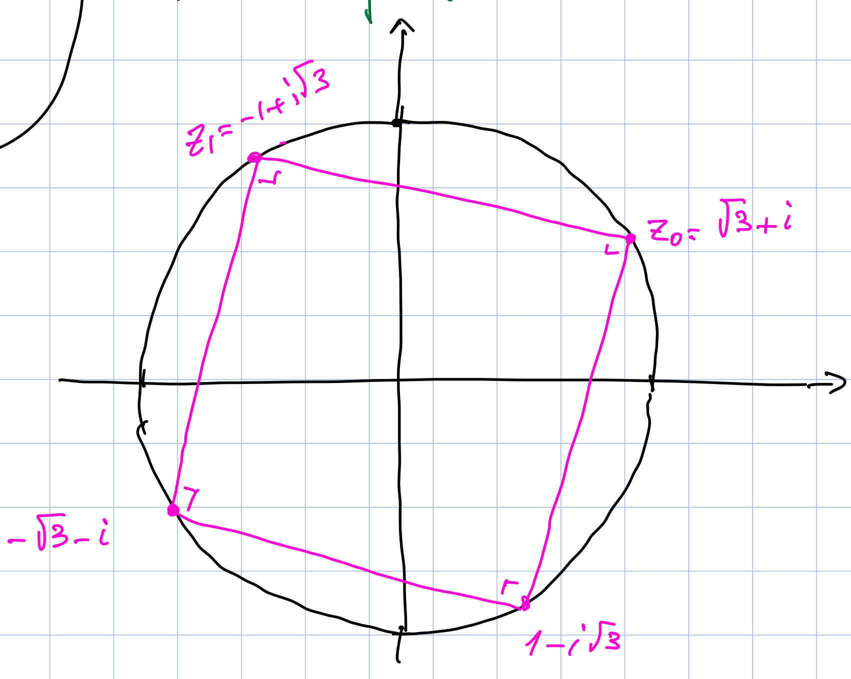
$$z_j = \sqrt[4]{|a|}\cdot e^{\left(\arg(a)/4 + \frac{2j\pi}{4}\right)\cdot i} \qquad j=0,1,2,3$$

$$|a| = 8\cdot\sqrt{1+3} = 16 \qquad \sqrt[4]{|a|} = 2$$



$$\arg(a) = \frac{2\pi}{3}$$

$$z_j = 2\cdot e^{i\cdot\left(\frac{\pi}{6}+j\cdot\frac{\pi}{2}\right)}$$



(f) $3(1+i)z^3 - 4(3+i)z^2 + (13-i)z + 2(i-2)$

Proviamo con $z=1$:

$$\cancel{3}+\cancel{3i}-\cancel{12}-\cancel{4i}+\cancel{13}-\cancel{i}+\cancel{2i}-\cancel{4}$$

$$
\begin{array}{c|cccc}
 & 3+3i & -12-4i & 13-i & -4+2i \\
1 & & 3+3i & -9-i & 4-2i \\
\hline
 & 3+3i & -9-i & 4-2i &
\end{array}
$$

$z_1 = 1$

$$z_{2,3} = \frac{9+i \pm \sqrt{\Delta}}{6(1+i)}$$

$$
\begin{aligned}
\Delta &= 81+18i-1-24(1+i)(2-i) \\
&= 80+18i-48-48i+24i-24 \\
&= 8-6i
\end{aligned}
$$

$\sqrt{\Delta} = \pm(a+ib)$

$$\begin{cases} a^2-b^2=8 \\ ab=-3 \end{cases} \qquad \begin{array}{l} a=3 \\ b=-1 \end{array}$$

$$z_{2,3} = \frac{\left((9+i)\pm(3-i)\right)(1-i)}{12} = \begin{cases} \dots \\ \dots \end{cases}$$