Ist. Mat. I - CIA
27/10/23

Foglio 3

$\boxed{14}$ sapendo una radice di $p(z)$ trovare le altre

(c) $4z^3 + (8i-4)z^2 - (5+4i)z + 1-i \qquad z_1 = -\frac{i}{2}$

$$
\begin{array}{c|ccc|c}
 & 4 & 8i-4 & -5-4i & 1-i \\
-i/2 & & -2i & 3+2i & i-1 \\
\hline
 & 4 & 6i-4 & -2-2i & /
\end{array}
$$

$p(z) = \left(z + \frac{i}{2}\right)\left(\;\dots\; \right)$

$= (2z+i)\left(2z^3 + (3i-2)z - (1+i)\right)$

$$
\begin{array}{c|cc|c}
 & 2 & 3i-2 & -1-i \\
-i/2 & & -i & 1+i \\
\hline
 & 2 & 2i-2 & /
\end{array}
$$

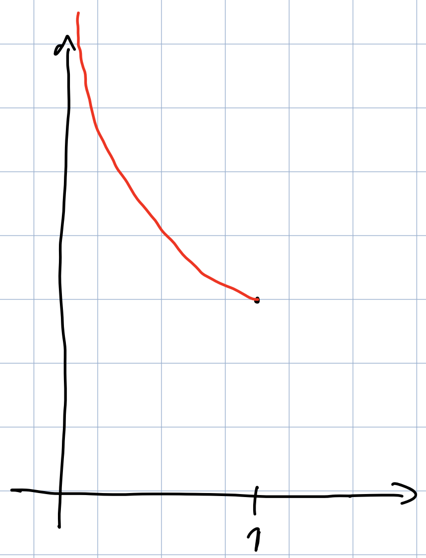
$p(z) = (2z+i)^2 \cdot (z + i - 1)$

$z_1 = -i/2 \qquad$ molt $= 2$

$z_2 = 1-i \qquad$ molt $= 1$

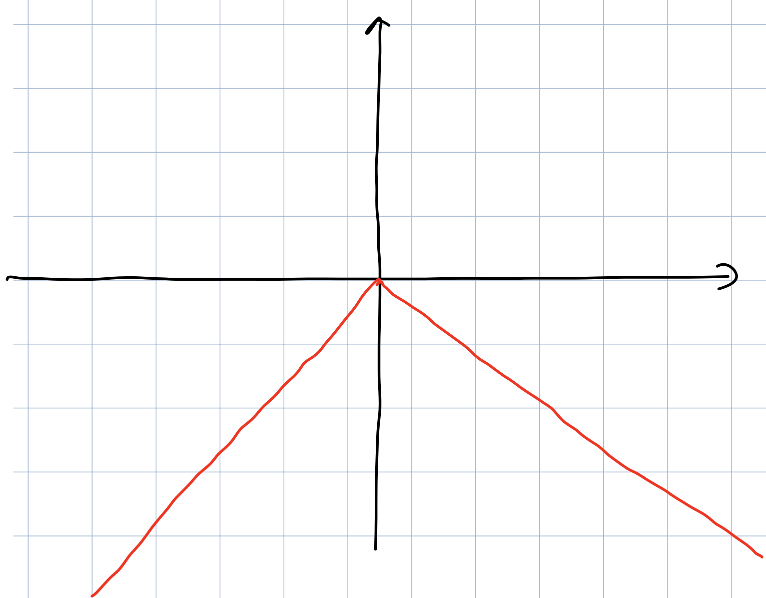
$\boxed{5}$ dire se f è lim sup/inf, se ha max/min
se è pari/dispari, crescente/dec, periodica.

(a) $f : (0,1] \to \mathbb{R} \qquad f(x) = \frac{1}{x}$



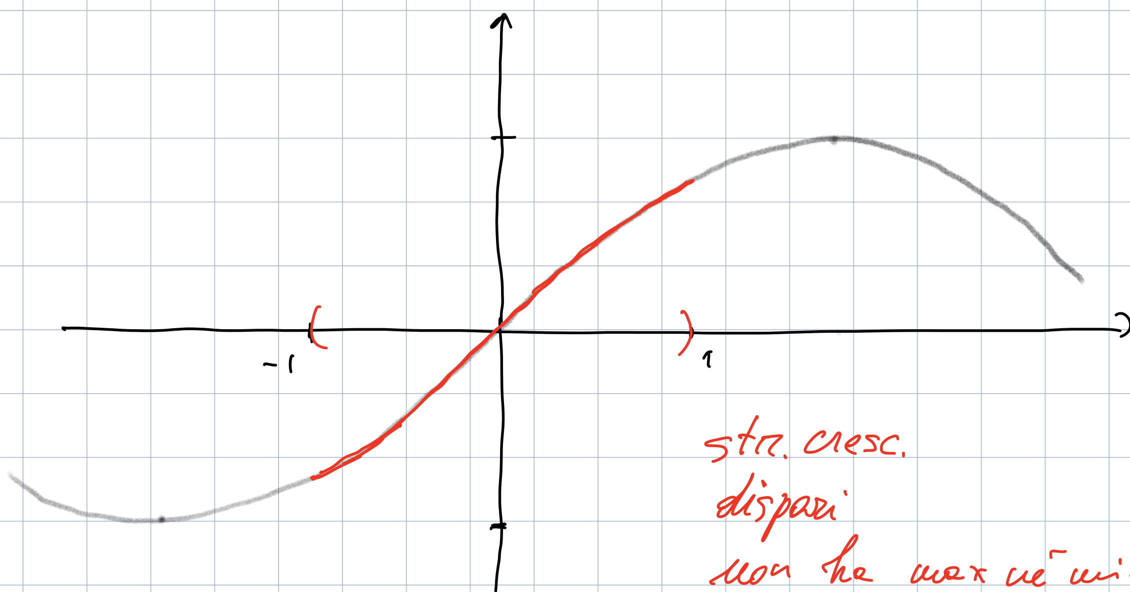
dec.
non sup. lim
min $= 1$

(b) $f : \mathbb{R} \to \mathbb{R} \qquad f(x) = -|x|$



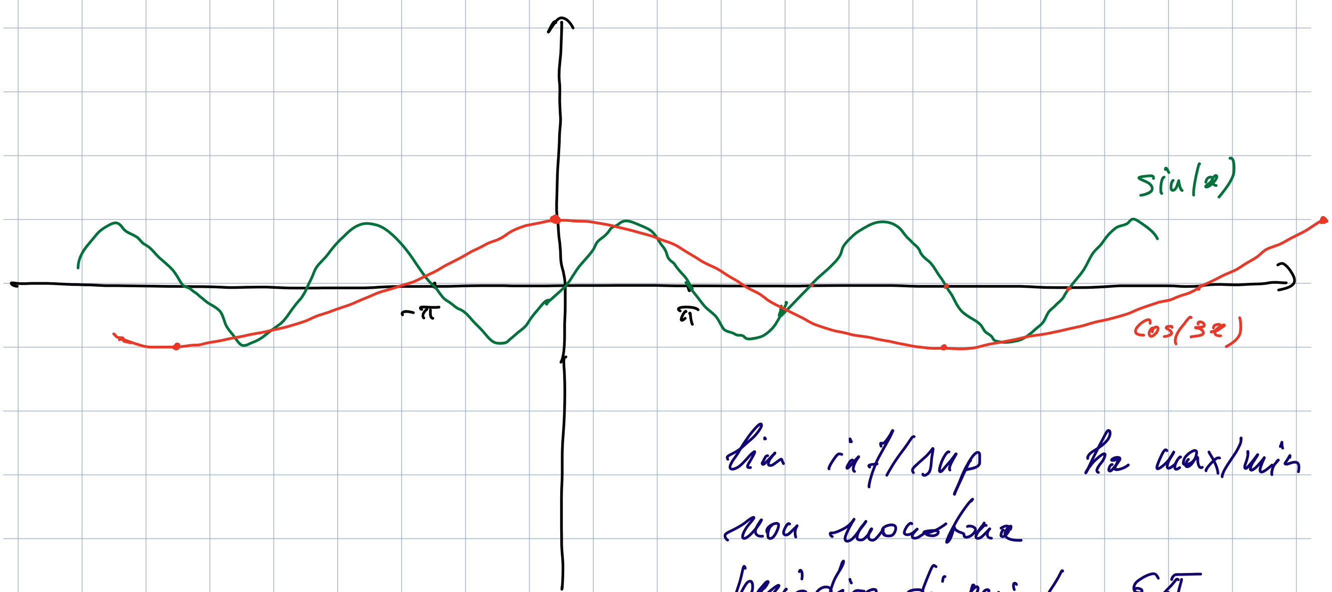
non inf lim
max $= 0$
pari
non monotona
(crescente o dec)

(c) $f: (-1,1) \to \mathbb{R}$ $\quad f(x) = \sin(x)$



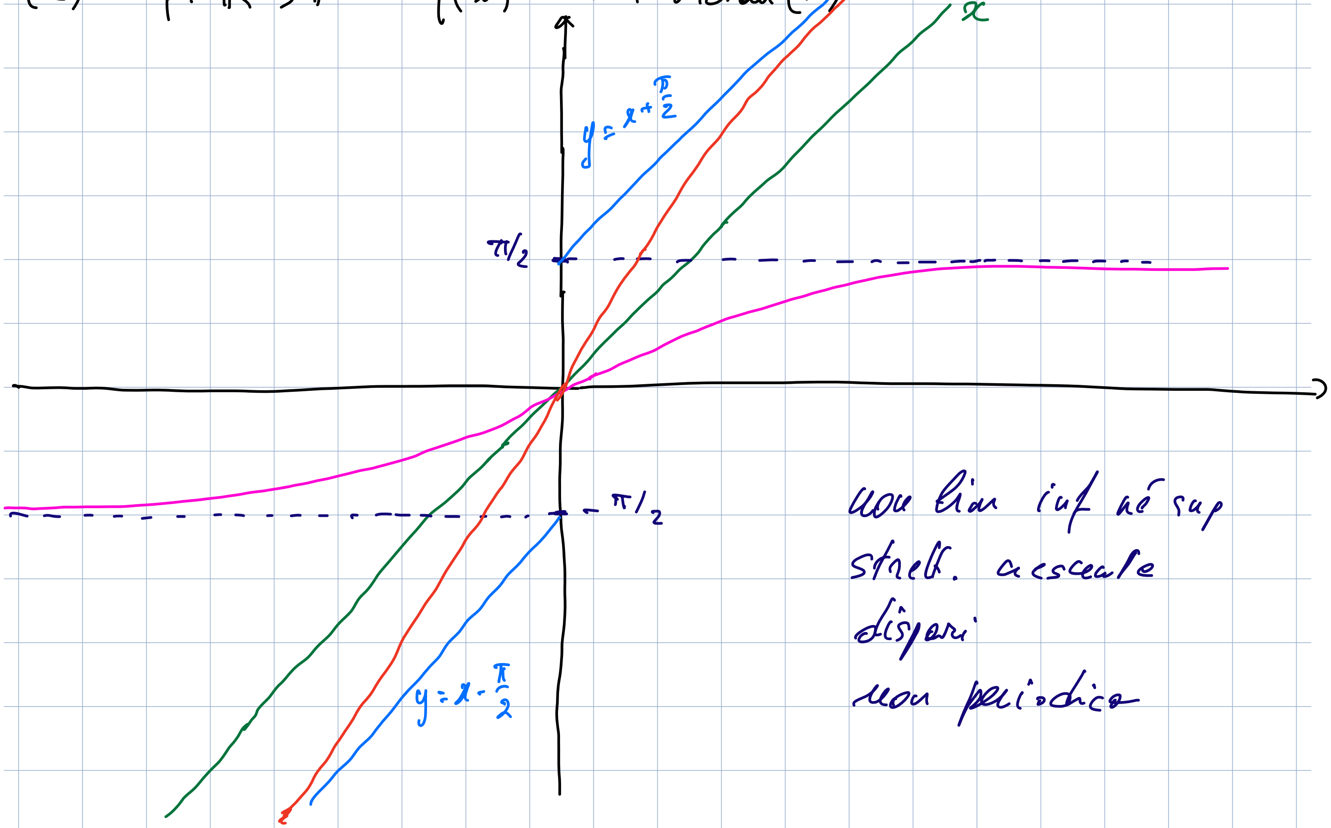
str. cresc.
dispari
non ha max né min
sup/inf lim

(d) $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \sin(x) + \cos\left(\frac{x}{3}\right)$



lim inf/sup ha max/min
non monotona
periodica di periodo 6π
né pari né dispari

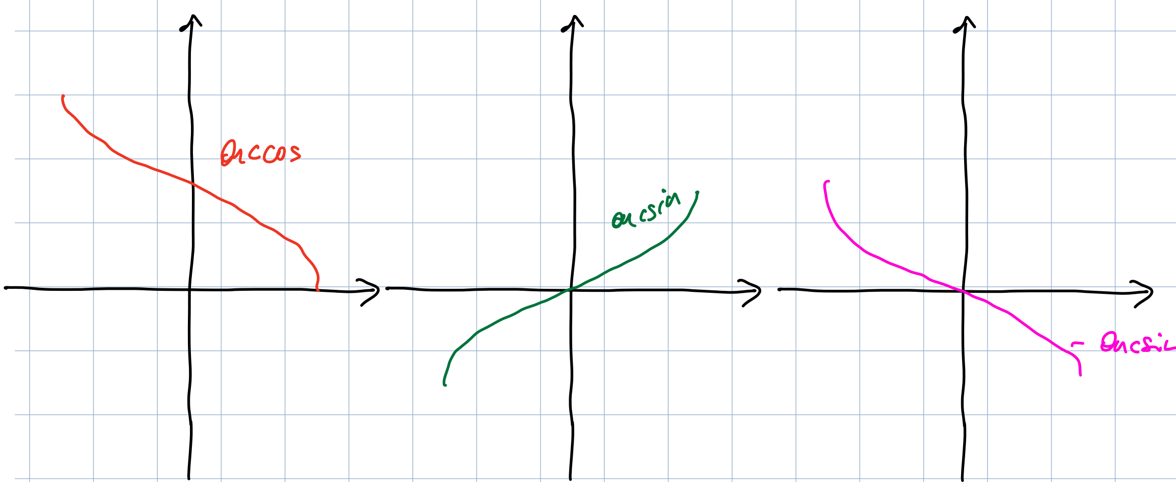
(e) $f: \mathbb{R} \to \mathbb{R} \quad f(x) = x + \arctan(x)$

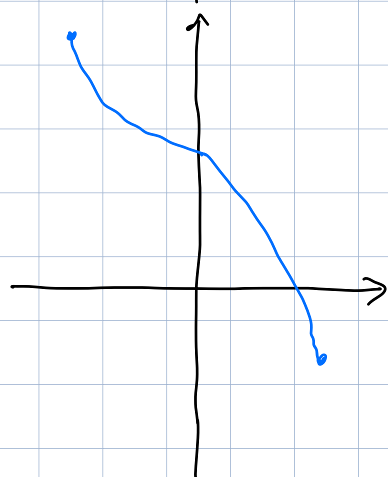


non lim inf né sup
strett. crescente
dispari
non periodica

(f) $\cosh(x) + \sinh(x) = e^x$

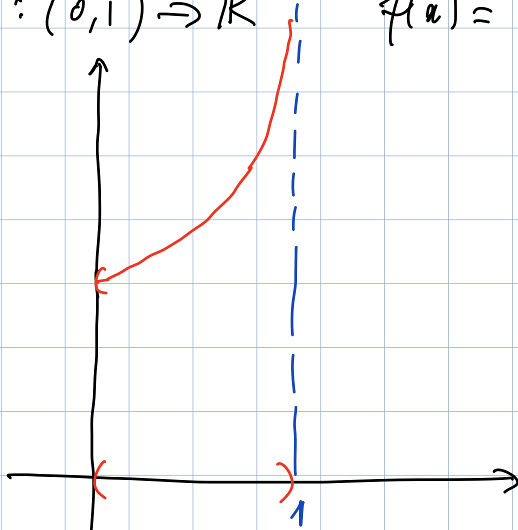
(g) $f:[-1,1]\to\mathbb{R}$ $\quad f(x)=\arccos(x)-\arcsin(x)$





ha max/min
str. decr.
né pari né dispari
non periodica

(h) $f:(0,1)\to\mathbb{R}$ $\quad f(x)=\dfrac{1}{1-x}$



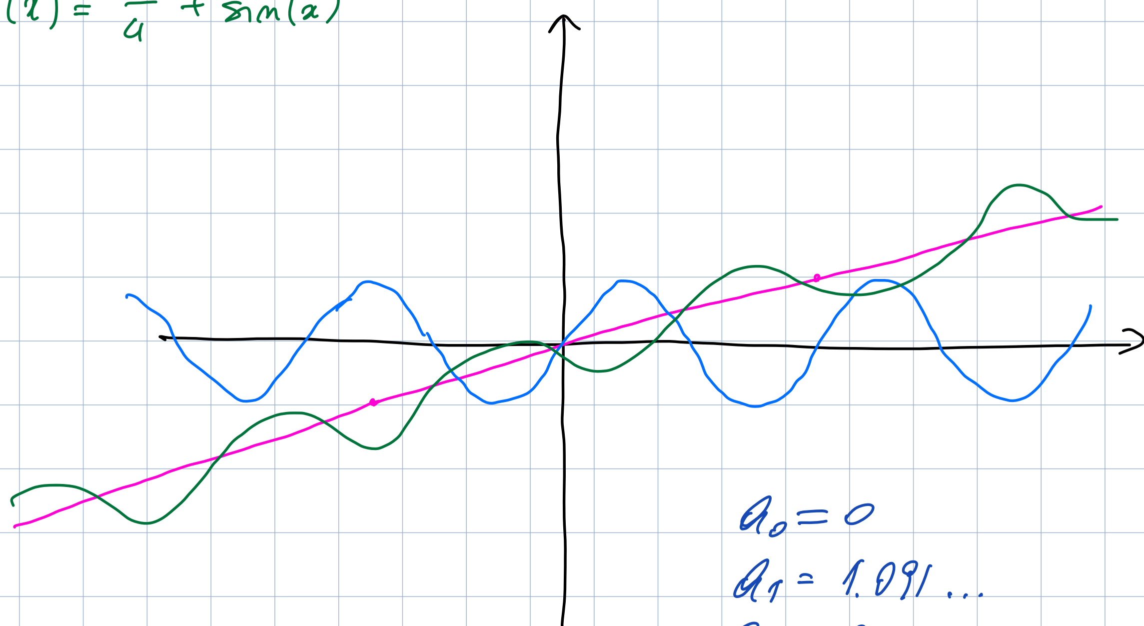
strett. crescente
non sup. lim.
inf lim. ma non ha min

$\boxed{6}$ a_n crescente?

(a) $$a_n = \underbrace{\underbrace{n^3}_{\text{cresc.}} + \underbrace{4}_{\text{pos.}}\underbrace{\sqrt{n}}_{\text{cresc}} \underbrace{- 2}_{\text{const}}}_{\text{crescente}}$$

(b) $$a_n = \frac{n}{4} + \sin(n)$$

$$f(x) = \frac{x}{4} + \sin(x)$$



$a_0 = 0$

$a_1 = 1.091\ldots$

$a_2 = 1.409\ldots$

$a_3 = 0.891\ldots$

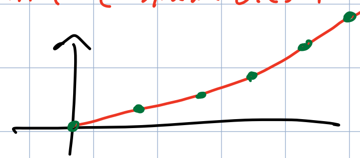
nem cresc. nem decr.

(c) $$a_n = n \cdot \arctan(n)$$

$$f : [0, +\infty) \to \mathbb{R} \qquad f(x) = \underbrace{x}\cdot\underbrace{\arctan(x)}$$

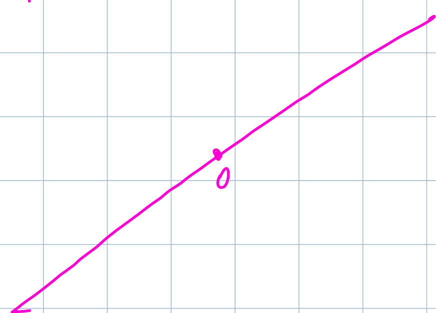
$\Longrightarrow (a_n)$ crescente

positive e strett. cresc.

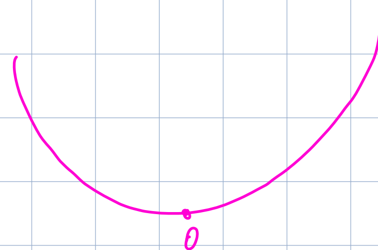


Oss: il prodotto di funz. crescenti non lo è se non sono positive

$f(x) = x$



$g(x) = x \cdot x$



$\boxed{7}$ provare $\lim (a_n) = L$ usando la def.

(a) $\lim\limits_{n \to +\infty} \dfrac{n^2 - 3n + 3}{n^2 + 4n + 3} = 1$

Dato $\varepsilon > 0$ qualsiasi voglio N t.c. $|a_n - 1| < \varepsilon \ \forall n \geq N$.

$$|a_n - 1| = \left| \frac{n^2 - 3n + 3}{n^2 + 4n + 3} - 1 \right| = \left| \frac{7n}{n^2 + 4n + 3} \right|$$

$$= \frac{7n}{n^2 + 4n + 3} < \frac{7n}{n^2} = \frac{7}{n}$$

Prendo allora $N > \dfrac{7}{\varepsilon}$ $\quad \left(N = \left[\dfrac{7}{\varepsilon} \right] + 1 \right)$

infatti se $n \geq N$ ho $n > \frac{7}{\varepsilon} \Rightarrow \frac{7}{n} < \varepsilon$

$\Rightarrow |a_n - 1| < \varepsilon$

(b) $\lim_{n \to +\infty} \sqrt{n} = +\infty$

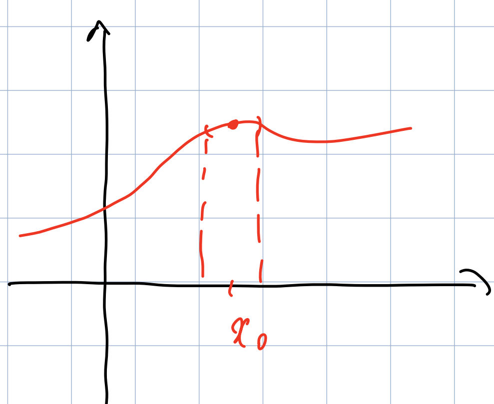
Dato $K > 0$ arbitrario cerco N t.c. $\sqrt{n} > K$ $\forall n \geq N$

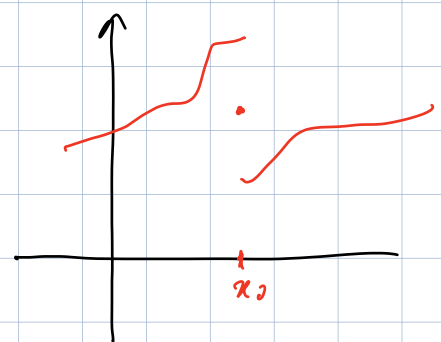
$\sqrt{n} > K \Leftrightarrow n > K^2$; basta prendere $N > K^2$.

"I limiti di somma, prodotto, quoziente, esponenziale, logaritmo ... si calcolano in modo naturale tranne se emerge una delle forme indeterminate:

$\infty - \infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, 1^∞, 0^0, ∞^0 "

Def: data $f: I \to \mathbb{R}$ e $x_0 \in I$ dico che f è continua in x_0 se $\lim_{x \to x_0} f(x) = f(x_0)$





<u>Fatto</u>: tutte le funzioni elementari
polinomi, razionali, trigonometriche, exp, log
sono continue in tutti i punti dove sono definite.

<u>Def</u>: $f: I \to \mathbb{R}$ è continua se è continua in tutti i punti di I.

Dimo della cont. di x^n (da cui la continuità di polinomi e razionali)

Devo vedere che $\forall x_0 \in \mathbb{R}$, $\lim_{x \to x_0} x^n = x_0^n$ cioè

$\lim_{h \to 0} (x_0+h)^n = x_0^n$; suppongo $|h| < 1$

$$\left| (x_0+h)^n - x_0^n \right| = \left| \sum_{k=0}^{n} \binom{n}{k} x_0^{n-k} \cdot h^k - x_0^n \right|$$

$$= \left| \cancel{x_0^n} + \sum_{k=1}^{n} \binom{n}{k} x_0^{n-k} \cdot h^k - \cancel{x_0^n} \right|$$

$$= \left| h \cdot \sum_{k=1}^{n} \binom{n}{k} x_0^{n-k} \cdot h^{k-1} \right|$$

$$= |h| \cdot \left| \quad '' \quad \right|$$

$$\leq |h| \cdot \left(\sum_{k=1}^{n} \binom{n}{k} \cdot |x_0|^{n-k} \cdot |h|^{k-1} \right)$$

$$\leq |h| \cdot \underbrace{\left(\sum_{k=1}^{n} \binom{n}{k} |x_0|^{n-k} \right)}_{\text{numero } A \text{ indip. da } h.}$$

Dato $\varepsilon > 0$ se $\delta = \frac{\varepsilon}{A}$ ho che per $|h| < \delta$

$$\left| (x_0 + h)^n - x_0^n \right| \leq A \cdot |h| < A \cdot \frac{\varepsilon}{A} < \varepsilon. \quad \blacksquare$$

Dimo continuità di sin (da cui cos, tan, cot).

visto $\lim_{x \to 0} \sin(x) = 0 = \sin(0)$.

Affermo che $\lim_{x \to 0} \cos(x) = 1$.

- Dando per buona continuità di $\sqrt{\ }$:

$$\cos(x) = \underbrace{\sqrt{\underbrace{1 - \underbrace{\sin^2(x)}_{\downarrow\, 0}}_{1}}}_{1}$$

- Osservo che $|\sin(x)| + |\cos(x)| \geq 1$; equivale a

$$\left(\qquad \right)^2 \geq 1^2$$

$$\cancel{\sin^2(x)} + 2|\sin(x) \cdot \cos(x)| + \cancel{\cos^2(x)} \geq \cancel{1}. \quad \text{OK}$$

$$1 - |\sin(x)| \leq |\cos(x)| \leq 1$$

anzi $$\underbrace{1-\underbrace{|\underbrace{\sin(x)}_{\downarrow\, 0}|}_{0}}_{1} \le \cos(x) \le \underset{\downarrow\, 1}{1} \quad \text{in } [-\pi/2, \pi/2]$$

Continuità di sin: devo vedere che $\forall x_0 \in \mathbb{R}$ ho

$$\lim_{x\to x_0} \sin(x) = \sin(x_0) \quad \text{cioè}$$

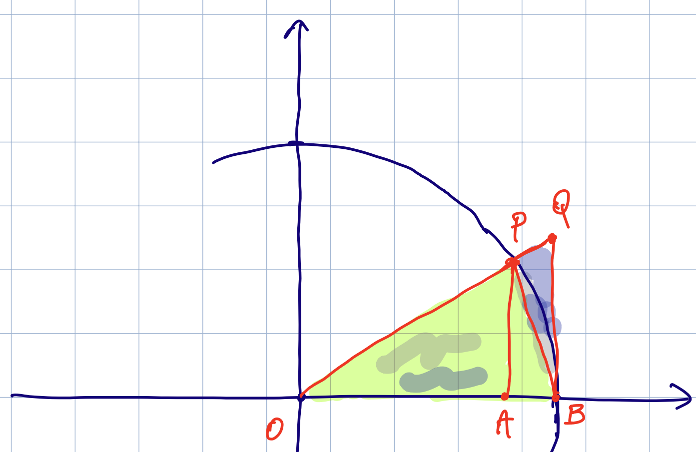
$$\lim_{h\to 0} \sin(x_0+h) = \sin(x_0)$$

$$\sin(x_0+h) = \underbrace{\underbrace{\sin(x_0)\cdot\underbrace{\cos(h)}_{\downarrow\, 1}}_{\sin(x_0)} + \underbrace{\cos(x_0)\cdot\underbrace{\sin(h)}_{\downarrow\, 0}}_{0}}_{\sin(x_0).}$$

$\blacksquare$

Prop: $$\lim_{x\to 0} \frac{\sin(x)}{x} = 1. \qquad \frac{0}{0}$$

Dim:



$P = \begin{pmatrix} \cos(x) \\ \sin(x) \end{pmatrix}$

$Q = \begin{pmatrix} 1 \\ tg(x) \end{pmatrix}$

Oss: $\widehat{OBP} \subset \widehat{OBP} \subset \widehat{OBQ}$

$\Rightarrow \quad \frac{1}{2}\sin(x) \leq \frac{1}{2}x \leq \frac{1}{2}tg(x)$

$\Rightarrow \quad \underset{\downarrow 1}{1} \leq \frac{x}{\sin(x)} \leq \underset{\downarrow 1}{\frac{1}{\cos(x)}}$

$\blacksquare$

Altri limiti notevoli:

- $\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$ $\qquad \frac{0}{0}$

- $\lim_{x\to \pm\infty} \left(1+\frac{1}{x}\right)^x = e$ $\qquad 0^{\pm\infty}$

- $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$ $\qquad \frac{0}{0}$

- $\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha$

So che $\lim_{y\to 0} \frac{\log(1+y)}{y} = 1$; sostituisco $y = (1+x)^\alpha - 1$ $(x\to 0 \Rightarrow y \to 0)$

$$\boxed{1} = \lim_{x\to 0} \frac{\log(1+(1+x)^\alpha - 1)}{(1+x)^\alpha - 1}$$

$$= \lim_{x\to 0} \frac{\alpha \cdot \log(1+x)}{(1+x)^\alpha - 1}$$

$$\boxed{= \alpha} \cdot \lim_{x\to 0} \boxed{\underbrace{\frac{\log(1+x)}{x}}_{\downarrow \, 1}} \cdot \frac{x}{(1+x)^\alpha - 1}$$

$$\Longrightarrow \lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha .$$

- $$\lim_{x\to 0} \frac{e^x - 1}{x} = 1 \qquad \frac{0}{0}$$

$x = \log(1+y)$ $\quad (x \to 0 \Longrightarrow y \to 0)$

$$\lim_{0} \frac{e^x - 1}{x} = \lim_{y\to 0} \frac{e^{\log(1+y)} - 1}{\log(1+y)} = \lim_{y\to 0} \frac{1+y-1}{\log(1+y)} = \lim_{y\to 0} \frac{y}{\log(1+y)}$$

$$\downarrow$$

$$1^{-1} = 1$$

Prop: la composizione di funzioni continue è continua

$$\lim_{x \to x_0} f(x) = f(x_0) \qquad \lim_{y \to f(x_0)} g(y) = g(f(x_0))$$

$$\Rightarrow \lim_{x \to x_0} g(f(x)) = g(f(x_0)).$$

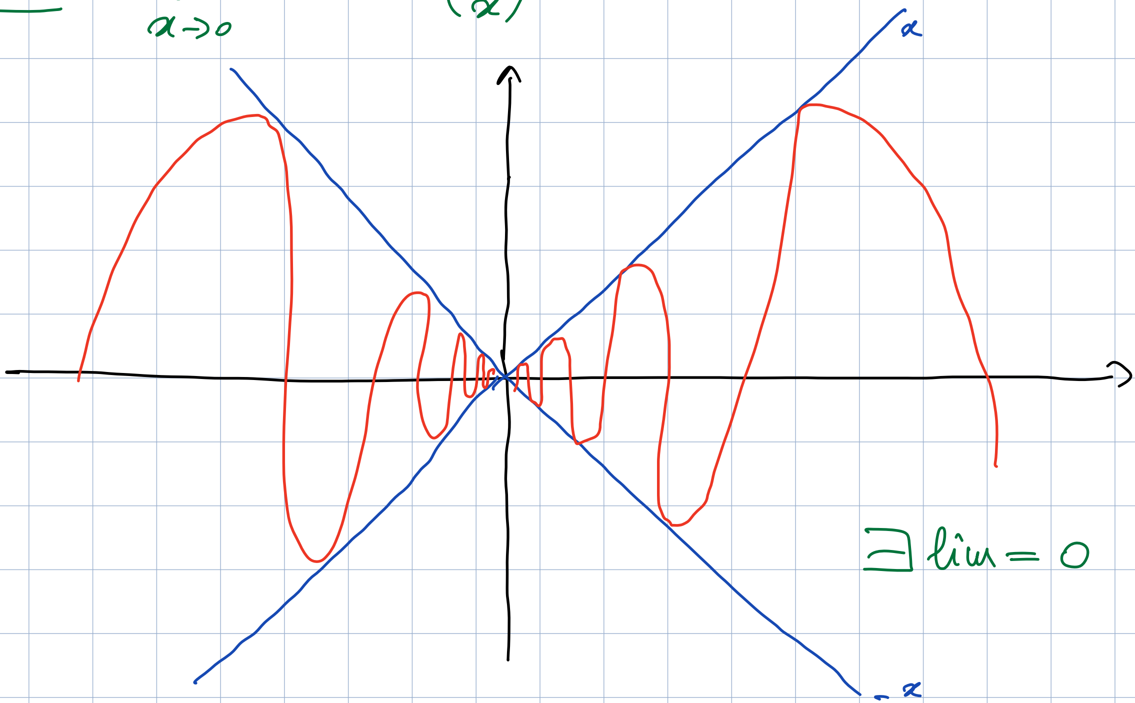
$\Rightarrow$ Tutte le funzioni che riesco a scrivere a partire da quelle elementari con operaz. algebriche + composizioni sono continue dove sono definite a meno che si divida per 0.

$\Rightarrow$ Tutti i limiti sono i valori se non si incontrano forme indet. o divisioni per 0
(forme indet: agli estremi dell'insieme di def.)

Ese: $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$

$\nexists$

Ese: $\lim_{x \to 0} x \cdot \sin\left(\frac{1}{x}\right)$



$\exists \lim = 0$