Ist. Mat. I - CIA
9/3/23

① (a) $A = \begin{pmatrix} 2 & -k & -1 \\ k+2 & 0 & 4 \\ -3 & k & 2 \end{pmatrix}$

A invert. $\Leftrightarrow \det(A) \neq 0$; in tal caso $A^{-1} = (b_{ij})$

$$b_{ij} = \frac{(-1)^{i+j}}{\det(A)} \cdot \det(A_{ji})$$

$$\det(A) = \det \begin{pmatrix} 2 & -k & -1 \\ k+2 & 0 & 4 \\ -1 & 0 & 1 \end{pmatrix} \quad R_3 + R_1$$

$\det(A) =$

$= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$

$= \sum_{i=1}^{n} (-1)^{j+i} a_{ji} \det(A_{ji})$

$$= (-1)^{1+2} \cdot (-k) \cdot \det \begin{pmatrix} k+2 & 4 \\ -1 & 1 \end{pmatrix}$$

$$= k \cdot (k+6)$$

$$A = \begin{pmatrix} 2 & -k & -1 \\ k+2 & 0 & 4 \\ -3 & k & 2 \end{pmatrix}$$

$\exists A^{-1}$ per $k \neq 0, -6$

$$A^{-1} = \frac{1}{k(k+6)} \cdot \begin{pmatrix} -4k & k & -4k \\ -2k-16 & * & * \\ * & * & * \end{pmatrix}$$

(b) $A: \begin{pmatrix} 1-k & 3k & -1 \\ 2k-2 & 0 & 2 \\ k-1 & 1 & 1 \end{pmatrix}$

$$\det \begin{pmatrix} 1+2k-3k^2 & 0 & -1-3k \\ 2k-2 & 0 & 2 \\ k-1 & 1 & 1 \end{pmatrix} \quad R_1 - 3k \cdot R_3$$

$$= -\det \begin{pmatrix} 1+2k-3k^2 & -1-3k \\ 2k-2 & 2 \end{pmatrix}$$

$$= -2 \cdot \det \begin{pmatrix} 1+2k-3k^2 & -1-3k \\ k-1 & 1 \end{pmatrix}$$

$$= -2 \cdot (\cancel{1} \cancel{+2k} \cancel{-3k^2} \cancel{+k} \cancel{+3k^2} \cancel{-1} \cancel{-3k}) \equiv 0$$

(c) $A: \begin{pmatrix} 1 & k & -1 \\ -k & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

$$\det(A) = \det \begin{pmatrix} 1 & k \\ -k & 1 \end{pmatrix} = 1 + k^2$$

$$A^{-1} = \frac{1}{1+k^2} \cdot \begin{pmatrix} 1 & -k & 2k+1 \\ k & * & * \\ * & * & * \end{pmatrix}$$

(e) $$\det \begin{pmatrix} a_{11} & a_{12} & \cdots & & a_{1m} \\ 0 & a_{22} & a_{23} & \cdots & a_{2m} \\ 0 & 0 & a_{33} & \cdots & a_{3m} \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & & 0 & a_{mm} \end{pmatrix}$$

$$= a_{11} \cdot \det \begin{pmatrix} a_{22} & a_{23} & \dots & a_{2m} \\ 0 & a_{33} & \dots & a_{3m} \\ & & \ddots & \\ 0 & \dots & 0 & a_{mm} \end{pmatrix}$$

$$= a_{11} \cdot a_{22} \cdot \dots \cdot a_{mm}$$

③ ⓐ $\det \begin{pmatrix} 1 & 0 & -1 \\ 3 & 2 & 3 \\ -1 & -1 & 1 \end{pmatrix} \begin{matrix} 1 & 0 \\ 3 & 2 \\ -1 & -1 \end{matrix}$

$$= 1 \cdot 2 \cdot 1 + 0 \cdot 3 \cdot (-1) + (-1) \cdot 3 \cdot (-1) - \left((-1) \cdot 2 \cdot (-1) + 1 \cdot 3 \cdot (-1) + 0 \cdot 3 \cdot 1 \right) = \dots$$

ⓑ $\det \begin{pmatrix} 0 & 2 & -1 & -3 \\ 4 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ 1 & 0 & -2 & 0 \end{pmatrix}$

$$= -(-3) \cdot \det \begin{pmatrix} 4 & 1 & 0 \\ 2 & -1 & 1 \\ 1 & 0 & -2 \end{pmatrix}$$

$$= 3 \cdot \det \begin{pmatrix} 4 & 1 & 0 \\ 6 & 0 & 1 \\ 1 & 0 & -2 \end{pmatrix} = 3 \cdot (-1) \cdot 1 \cdot \det \begin{pmatrix} 6 & 1 \\ 1 & -2 \end{pmatrix}$$

$$= 39$$

(c) $\det \begin{pmatrix} 5 & 2 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ 4 & 3 & -1 & 1 \\ 1 & 0 & 0 & 3 \end{pmatrix}$

$$= \det \begin{pmatrix} 5 & 7 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 4 & 7 & -1 & 1 \\ 1 & 1 & 0 & 3 \end{pmatrix}$$

$C_2 + C_1$

$$= -\det \begin{pmatrix} 7 & -1 & 1 \\ 7 & -1 & 1 \\ 1 & 0 & 3 \end{pmatrix} = 0$$

(d) $\det \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 3 & 2 & 1 & 1 \end{pmatrix}$

$$= \det \begin{pmatrix} -3 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ -1 & 2 & 1 & 1 \end{pmatrix} = -\det \begin{pmatrix} 3 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$C_1 - 2 \cdot C_2$

$$= -\det \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad R_1 - R_3 \qquad = -2$$

Un metodo per trovare l'inversa:

- $A \in \mathbb{R}^{n \times n}$ • $T = (A, I_n)$
- faccio operazioni $T \rightsquigarrow T'$ con operazioni sulle righe: sostituisco a una riga $\alpha \cdot$ lei $+ \beta \cdot$ un'altra; e (scambi di riga)

- mi riconduco a (I_n, B)
- Fatto: $B = A^{-1}$.

Perché funziona? Una operazione sulle righe $T \rightsquigarrow T'$ realizza $T' = M \cdot T$ con $M \in \mathcal{M}_{n \times n}$

$$T = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}$$

sostituire r_i con $\alpha r_i + \beta r_j$ si realizza con

$$\begin{array}{c c} & \\ i \rightarrow & \begin{pmatrix} 1 & 0 & \cdots & & & & & 0 \\ 0 & 1 & \cdots & & & & & 0 \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ & & & \alpha\, 0 \cdots 0 & \beta & 0 \cdots 0 & & \\ & 0 & & & 1 & & 0 & \\ & & & & & \ddots & & \\ & & & & & & & 1 \end{pmatrix} T \\ & \uparrow \\ & i \end{array}$$

Dunque $T = (A, I_n) \xrightarrow[k \text{ operaz}]{} (I_n, B)$

$$\Rightarrow (I_n, B) = \underbrace{(M_k \cdots M_3 \cdot M_2 \cdot M_1)}_{M} \cdot (A, I_n)$$

$$(I_n, B) = M \cdot (A, I_n)$$

$$\begin{cases} M \cdot A = I_n \\ M \cdot I_n = B \end{cases} \Rightarrow \begin{cases} M = A^{-1} \\ B = M \end{cases}$$

$$\Rightarrow B = A^{-1}$$

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 4 & 1 & 3 \\ -2 & 7 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 & -1 & 1 & 0 & 0 \\ 4 & 1 & 3 & 0 & 1 & 0 \\ -2 & 7 & 5 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{cl} \begin{pmatrix} 3 & 2 & -1 & 1 & 0 & 0 \\ 0 & -5 & 13 & -4 & 3 & 0 \\ 0 & 25 & 13 & 2 & 0 & 3 \end{pmatrix} & \begin{array}{l} \\ 3\pi_2 - 4\pi_1 \\ 3\pi_3 + 2\pi_1 \end{array} \end{array}$$

$$\begin{array}{cl} \begin{pmatrix} 3 & 2 & -1 & 1 & 0 & 0 \\ 0 & -5 & 13 & -4 & 3 & 0 \\ 0 & 0 & 78 & -18 & 15 & 3 \end{pmatrix} & \begin{array}{l} \\ \\ \pi_3 + 5\pi_2 \end{array} \end{array}$$

$$\begin{array}{cl} \begin{pmatrix} 3 & 2 & -1 & 1 & 0 & 0 \\ 0 & -5 & 13 & -4 & 3 & 0 \\ 0 & 0 & 26 & -6 & 5 & 1 \end{pmatrix} & \begin{array}{l} \\ 1/3 \cdot \pi_3 \\ \\ \end{array} \end{array}$$

$$\begin{array}{cl} \begin{pmatrix} 78 & 52 & 0 & 20 & 5 & 1 \\ 0 & -10 & 0 & -2 & 1 & -1 \\ 0 & 0 & 26 & -6 & 5 & 1 \end{pmatrix} & \begin{array}{l} 26\pi_1 + \pi_3 \\ 2\pi_2 - \pi_3 \\ \\ \end{array} \end{array}$$

$$\begin{array}{cl} \begin{pmatrix} 390 & 0 & 0 & 48 & 51 & -21 \\ 0 & -10 & 0 & -2 & 1 & -1 \\ 0 & 0 & 26 & -6 & 5 & 1 \end{pmatrix} & \begin{array}{l} 5\pi_1 + 26\pi_2 \\ \\ \\ \end{array} \end{array}$$

$$\begin{array}{cl} \begin{pmatrix} 1 & 0 & 0 & 8/65 & 17/130 & -7/130 \\ 0 & 1 & 0 & 1/5 & -1/10 & 1/10 \\ 0 & 0 & 1 & -3/13 & 5/26 & 1/26 \end{pmatrix} & \begin{array}{l} \pi_1/390 \\ \pi_2/-5 \\ \pi_3/26 \end{array} \end{array}$$

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & -ab+ab \\ cd-cd & -bc+ad \end{pmatrix}$$

$$= (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$