Ist. Mat. I – CIA
16/2/23

① $f:[a,b]\to\mathbb{R}$, $f\geq 0$ unif. cont $\Longrightarrow \exists \int_a^b f(x)\,dx$

② $f:[a,b]\to\mathbb{R}$ cont $\Longrightarrow$ unif. cont.

Def: data $\{x_n\}_{n=0}^{\infty}$ chiamo $\{x'_k\}_{k=0}^{\infty}$ estratta se è ottenuta da $\{x_n\}$ cancellando alcuni termini.

$\cancel{\sqrt{2}},\ 3,\ \cancel{4},\ \cancel{-7},\ 1/3,\ \times,\ \times,\ \times,\ \cdot,\ \cdot,\ \times,\ \times,\ \times,\ \times,\ \cdot,\ \cdot,\ \cdot,\ \cdot,\ \cdot,$

Lem: se $\{x_n\}\subset[a,b]$ allora ha una estratta che converge a $\bar{x}\in[a,b]$.

"Dim" Divido a metà $[a,b]$; almeno una delle due metà contiene infiniti x_n; tengo lei e continuo; scelgo x'_k nel k-esimo intervallo della succ. diverso dai precedenti:

$\bigcap$ tutti intervallini $=\{\bar{x}\}$, $x'_k\to\bar{x}$. □

Dimo (2): f continua su $[a,b] \Rightarrow$ unif. cont.

Devo vedere: $\forall \varepsilon > 0 \ \exists \delta$ t.c. se $|x-y| < \delta$ allora $|f(x) - f(y)| < \varepsilon$.

Assurdo: $\exists \varepsilon > 0$ t.c. $\forall \delta > 0 \ \exists \ x, y \in [a,b]$ con $|x-y| < \delta$ ma $|f(x) - f(y)| \geq \varepsilon$

Uso questo con $\delta = \frac{1}{n}$ trovo x_n, y_n con $|x_n - y_n| < \frac{1}{n}$ $\quad |f(x_n) - f(y_n)| \geq \varepsilon$.

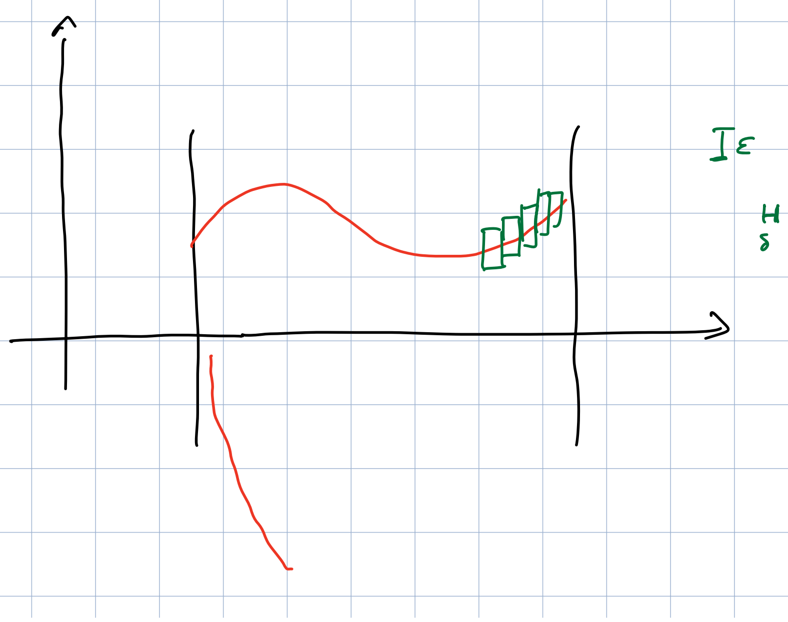
Estraggo $x_k' \to \bar{x}$; $\qquad x_n - y_n \to 0$

$\Rightarrow x_k' - y_k' \to 0$

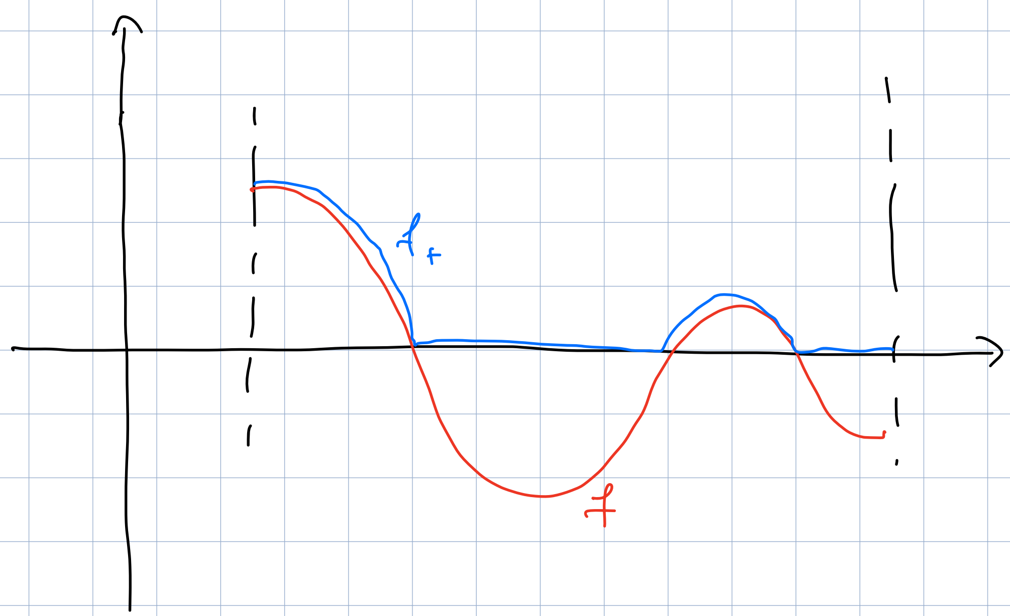
$\Rightarrow y_k' \to \bar{x}$

$x_k', y_k' \to \bar{x} \Rightarrow f(x_k') - f(y_k') \to 0$ (f continua)

assurdo poiché $|f(x_k') - f(y_k')| \geq \varepsilon$. $\boxtimes$



③ f cont $\Longrightarrow$ $f_\pm$ continue



Esercizio: $\left| f_\pm(x) - f_\pm(y) \right| \leq \left| f(x) - f(y) \right|$

(distinguere i segni di $f(x)$ e $f(y)$)

---

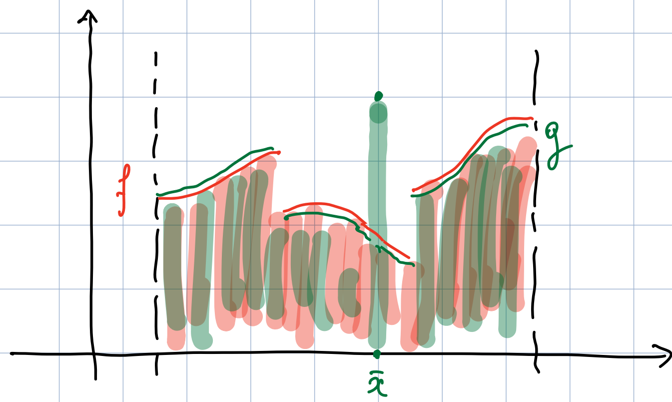
f continua su $[a,b] \Longrightarrow \int_a^b f(x)\,dx$.

Prop: $f : [a,b] \to \mathbb{R}$, $\exists \int_a^b f(x)\,dx$.

$g : [a,b] \to \mathbb{R}$ t.c. $g(x) = f(x) \quad \forall x \neq \bar{x}$

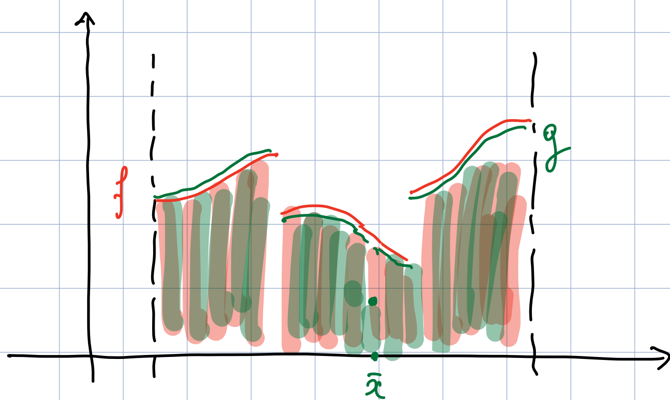
$\Longrightarrow \exists \int_a^b g(x)\,dx = \int_a^b f(x)\,dx$.

Dim: basta farlo con $f, g \geq 0$



Se $P_- \le f$, $P_- \le g$;
Se $P_+ \ge f$ aggiungo un rettangolo con base piccola e ho $P_+' \ge g$

$A(P_+') - A(P_-) < \varepsilon$



Se $P_+ \ge f$, $P_+ \ge g$.
Se $P_- \le f$ tolgo rettangolo con base piccola e ho $P_-' \le g$

$A(P_+) - A(P_-') < \varepsilon$

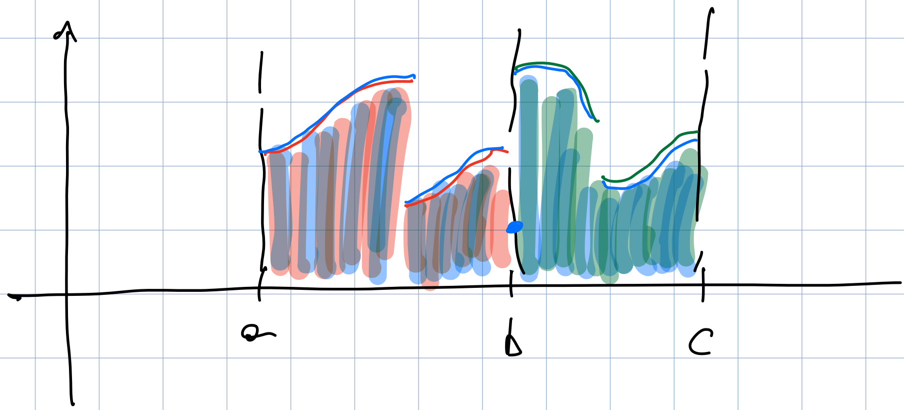
$\blacksquare$

Prop: date $f : [a,b] \to \mathbb{R}$, $\quad \exists \int_a^b f(x)\,dx$

$g : [b,c] \to \mathbb{R}$, $\quad \exists \int_b^c g(x)\,dx$

$$h : [a,c] \to \mathbb{R} \qquad h(x) = \begin{cases} f(x) & x \in [a,b) \\ \text{a caso} & x = b \\ g(x) & x \in (b,c] \end{cases}$$

$$\Rightarrow \exists \int_a^c h(x)\,dx = \int_a^b f(x)\,dx + \int_b^c g(x)\,dx$$



se $f,g \geq 0$

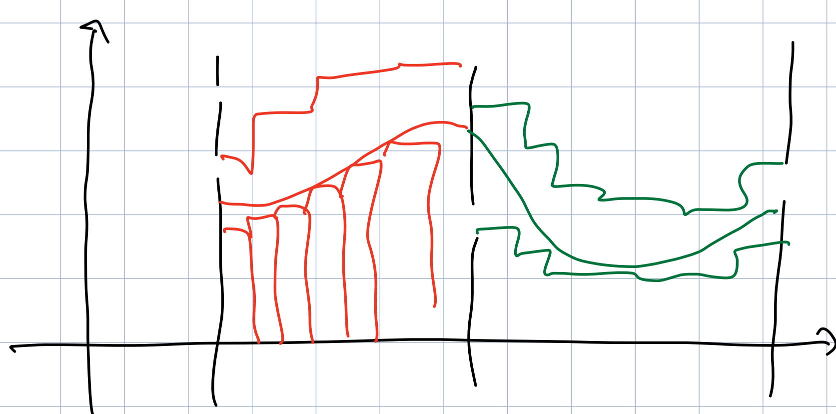
Dimo: posso supporre $f(b)=g(b)=h(b)$.

Se ho plurirettangoli $P_- \leq f \leq P_+$ $\quad A(P_+)-A(P_-)<\varepsilon/2$

$Q_- \leq g \leq Q_+$ $\quad A(Q_+)-A(Q_-)<\varepsilon/2$

$R_\pm = P_\pm \cup Q_\pm$ $\qquad R_- \leq f+g \leq R_+$

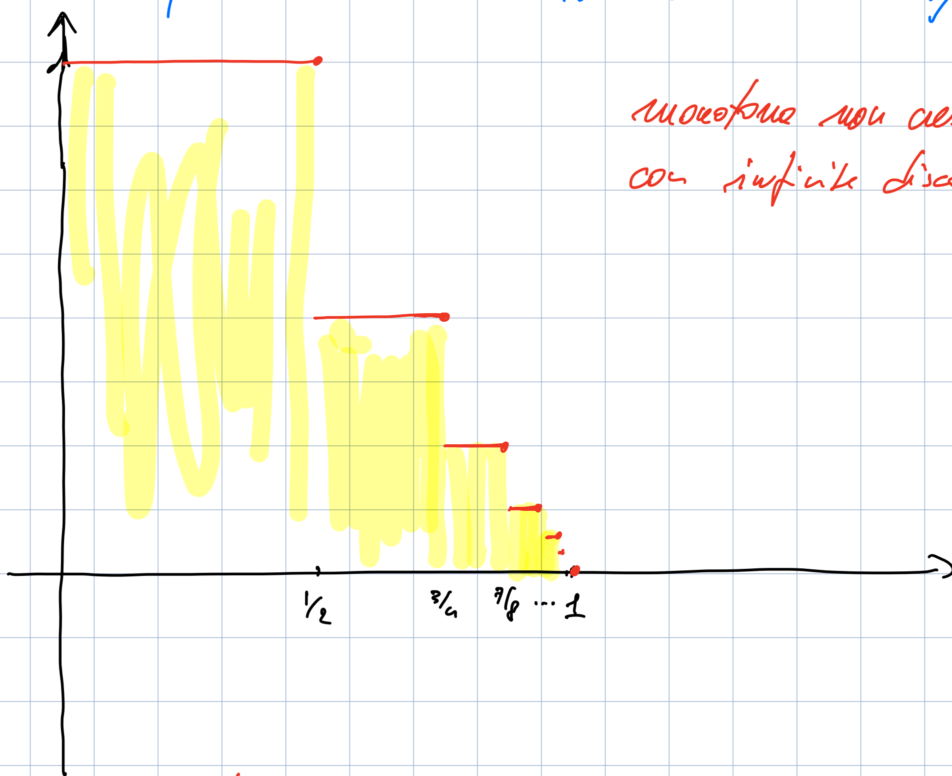
$A(R_+)-A(R_-) < \varepsilon/2+\varepsilon/2 = \varepsilon.$



Caso f,g che cambiano segno... $\boxtimes$

Fatto: una funzione limitata su $[a,b]$ con un numero finito di discontinuità è integrabile.

Teo: una funzione monotona su $[a,b]$ è integrabile.



monotona non crescente
con infinite discontinuità

Area sottografico = $\sum$ serie geom. di ragione $1/4$

Prop: se $f, g : [a,b] \to \mathbb{R}$, $k, h \in \mathbb{R}$, $\exists \int_a^b f(x)\,dx, \int_a^b g(x)\,dx$

$$\Rightarrow \exists \int_a^b \big(k \cdot f(x) + h \cdot g(x)\big)\,dx$$

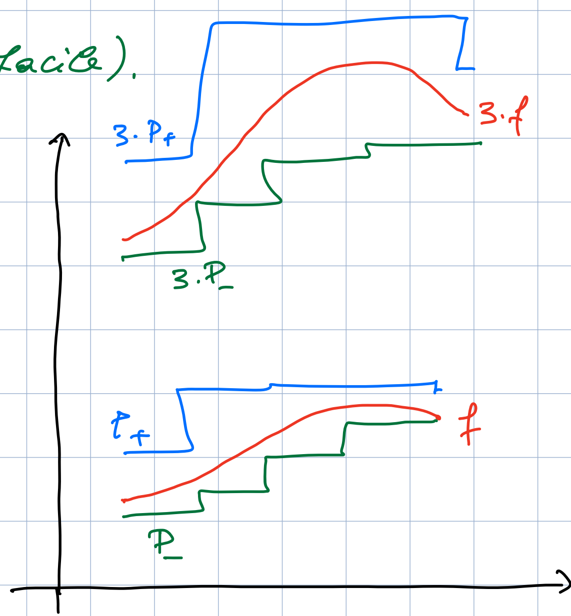
$$= k \cdot \int_a^b f(x)\,dx + h \cdot \int_a^b g(x)\,dx.$$

(l'integrale è <u>lineare</u> : omogeneo $\int k \cdot f = k \cdot \int f$

addittivo $\quad \int(f+g) = \int f + \int g \quad)$

"Dimo": omogeneo (facile).

$k>0,\ f\geq 0$

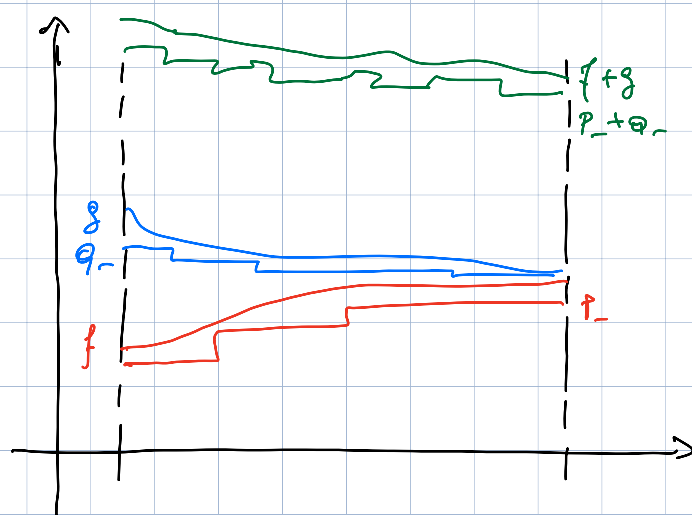


$k>0$, f segno qualsiasi ; $(k \cdot f)_\pm = k \cdot f_\pm$

$k<0$: ribaltare i plurirettangoli.

addittivo: $\displaystyle\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$

Caso $f\geq 0,\ g\geq 0$



Se $\quad P_- \leq f \leq P_+ \qquad \mathcal{A}(P_+) - \mathcal{A}(P_-) < \varepsilon/2$

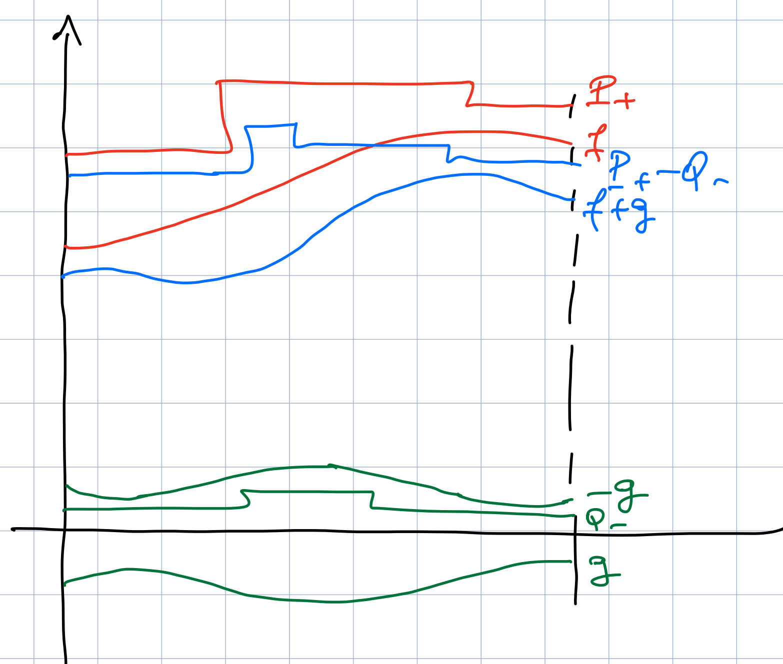
$\qquad\quad Q_- \leq g \leq Q_+ \qquad \mathcal{A}(Q_+) - \mathcal{A}(Q_-) < \varepsilon/2$

$$P_- + Q_- \leq f + g \leq P_+ + Q_+$$

$$\mathcal{A}(P_+ + Q_+) - \mathcal{A}(P_- + Q_-) < \varepsilon/2 + \varepsilon/2 = \varepsilon .$$

<u>Altro caso</u>

$f \geq 0 \qquad g \leq 0 \qquad f \geq -g \quad (\text{cioè } f+g \geq 0)$



$P_- \leq f \leq P_+ \qquad Q_- \leq -g \leq Q_+ \qquad Q_+ \leq P_-$

$$P_- - Q_+ \leq f + g \leq P_+ - Q_-$$

$\int f \qquad \int -g \qquad\qquad \int f \qquad \int -g$

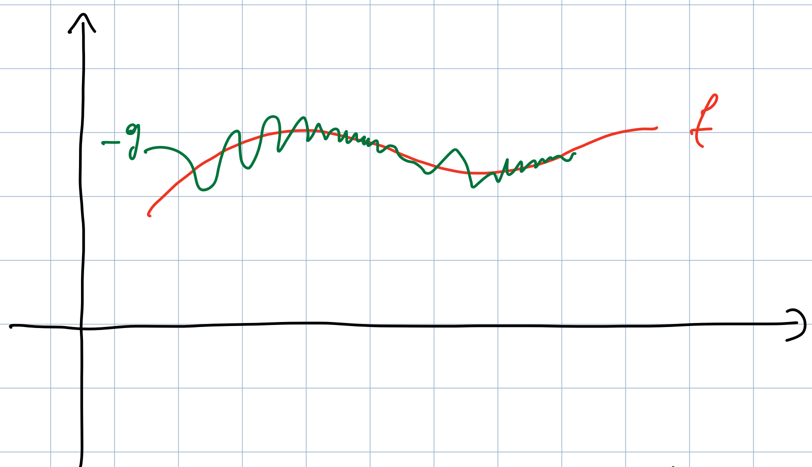
$\int g \qquad\qquad\qquad\qquad \int g$

Idee: 1) trattare tutti i casi

$f \geq 0 \quad g \geq 0 \qquad f \gtrless -g$

2) suddividere $[a,b]$ in modo che su ogni suddivisione ci sia uno di questi casi.

Sottigliezza: posso avere $f+g=0$ in infiniti punti



$g(x) = -f(x) + x \cdot \sin\left(\frac{1}{x}\right)$ su $[0,1]$

$\blacksquare$

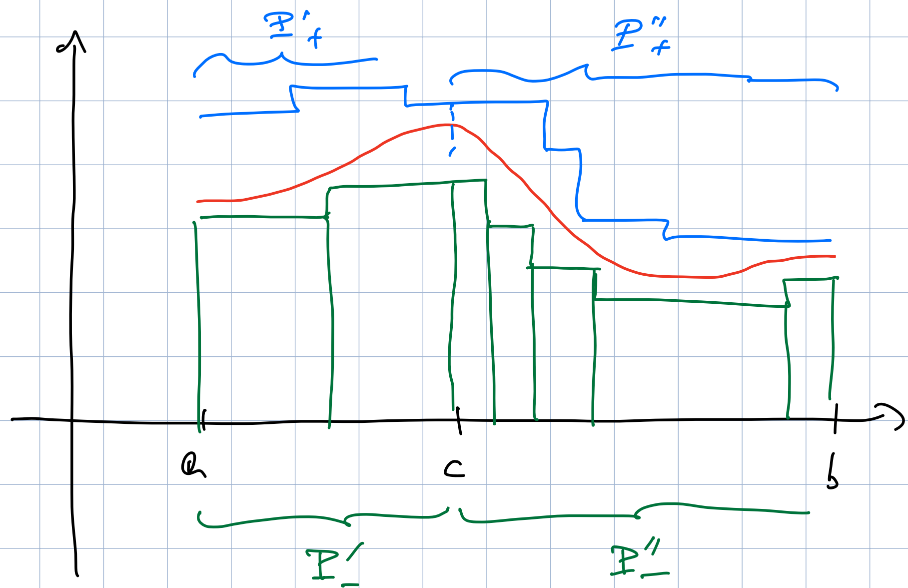
Prop: se $f : [a,b] \to \mathbb{R}$, $\exists \int_a^b f(x)\,dx$
$a < c < b$

$\Rightarrow \quad \exists \int_a^c f(x)\,dx, \quad \int_c^b f(x)\,dx, \quad \int_a^c \dots + \int_c^b \dots = \int_a^b f(x)\,dx$

Dim: basta farlo per $f_\pm$ cioè supporre $f \geq 0$.

Dati $P_- \leq f \leq P_+$ posso supporre che c sia uno dei punti di suddivisione

$\longrightarrow$ posso separare $P_\pm = P'_\pm \cup P''_\pm$ ...



Oss: $f \geq 0 \Longrightarrow \int_a^b f(x)\,dx \geq 0$

Cor: $f \geq g \Longrightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

(usando la linearità)

Prop: se $\exists \int_a^b f(x)\,dx$, $\exists \int_a^b |f(x)|\,dx$

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx.$$

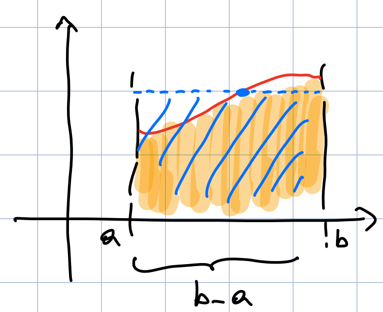
Dim: $f = f_+ - f_-$ $\qquad |f| = f_+ + f_-$

$\Rightarrow \exists \int |f| = \int f_+ + \underbrace{\int f_-}_{\geq 0}$

$$\int f = \int f_+ - \int f_-$$ $\boxtimes$

<u>Teo della media integrale</u>: data $f : [a,b] \to \mathbb{R}$ continua

$$\exists c \in [a,b] \text{ t.c. } \underbrace{\frac{1}{b-a} \cdot \int_a^b f(x)\,dx}_{} = f(c).$$



<u>Dimo</u>: $m = \min_{[a,b]} f \qquad M = \max_{[a,b]} f$

$$m \le f \le M$$

$$\int_a^b m \le \int_a^b f \le \int_a^b M$$

$$\int_a^b m = (b-a)\cdot m \qquad \int_a^b M = (b-a)\cdot M$$

$$\Longrightarrow \quad m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M$$

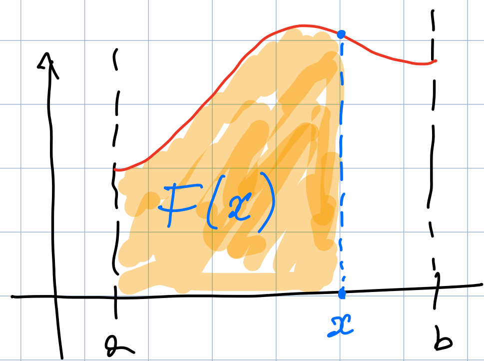
↳ compreso fra min e max

$\Longrightarrow$ è assunto in qualche $c \in [a,b]$

Teo fondamentale del calcolo integrale:

$f : [a,b] \to \mathbb{R}$ continua

$$F(x) = \int_a^x f(t)\,dt$$



$$\Rightarrow \exists F'(x) = f(x).$$

Dimo: $\displaystyle \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$

$h > 0$: $$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \right)$$

$$= \frac{1}{h}\left(\cancel{\int_a^x f(t)\,dt} + \int_x^{x+h} f(t)\,dt - \cancel{\int_a^x f(t)\,dt} \right)$$

$$= \frac{1}{h}\int_x^{x+h} f(t)\,dt = \frac{1}{(x+h)-x}\int_x^{x+h} f(t)\,dt = f(c) \longrightarrow f(x)$$

$c \in [x, x+h]$

Teo precedente media integrale

$h < 0$ $$\frac{F(x+h) - F(x)}{h} = \frac{1}{-h}\left(F(x) - F(x+h) \right)$$

$$= \frac{1}{-h}\left(\int_a^x f(t)\,dt - \int_a^{x+h} f(t)\,dt\right)$$

$$= \frac{1}{-h}\left(\cancel{\int_a^{x+h}} + \int_{x+h}^{x} - \cancel{\int_a^{x+h}}\right)$$

$$= \frac{1}{-h}\int_{x+h}^{x} f(t)\,dt = \frac{1}{x-(x+h)}\int_{x+h}^{x} f(t)\,dt = f(c) \longrightarrow f(x)$$

$c \in [x+h, x]$

$\blacksquare$

<u>Cor</u>: data $f:[a,b]\to\mathbb{R}$ continua se ho $G:[a,b]\to\mathbb{R}$ t.c. $G' = f$ (G primitiva di f)

$$\Rightarrow \int_a^b f(x)\,dx = G(b) - G(a) \quad \left(= G(x)\Big|_a^b \right)$$

<u>Es</u>: $\displaystyle\int_{-3}^{7} 4x^3\,dx = x^4\Big|_{-3}^{7} = 7^4 - (-3)^4 = \dots$

<u>Dim</u>: $F(x) = \displaystyle\int_a^x f(t)\,dt$

- $F(a) = 0$
- $F(b) = \displaystyle\int_a^b f(x)\,dx$
- $F'(x) = f(x)$

$$G' = F' = f \Rightarrow (G-F)' = 0 \Rightarrow G - F \equiv c$$

$$G(b)-G(a) = (F(b)+\cancel{c}) - (F(a)+\cancel{c})$$

$$\underbrace{F(b)}_{=\int_a^b f(x)\,dx} \qquad \underbrace{F(a)}_{=0}$$

$\boxtimes$

<u>Def</u>: chiamo integrale indefinito (per f continua)

$\int f(x)\,dx$ = tutte le primitive di f.

$$\boxed{\int f(x)\,dx = G(x)+c \implies \int_a^b f(x)\,dx = G(x)\Big|_a^b}$$

cioè $G'(x) = f(x)$

Invertendo le regole di derivazione note:

- $\int x^\alpha\,dx = \frac{1}{\alpha+1}x^{\alpha+1} + c \qquad \alpha \neq -1$
- $\int \frac{1}{x}\,dx = \log|x| + c$
- $\int \sin(x)\,dx = -\cos(x) + c$
- $\int \cos(x)\,dx = \sin(x) + c$
- $\int \frac{1}{\cos^2(x)}\,dx = \tan(x) + c$

- $\int \frac{1}{\sin^2(x)}\,dx = -\cot(x) + c$

- $\int e^x\,dx = e^x + c$

- $\int a^x\,dx = \frac{1}{\log(a)} \cdot a^x + c$

- $\int \frac{1}{1+x^2}\,dx = \operatorname{atan}(x) + c$

- $\int \frac{1}{\sqrt{1-x^2}}\,dx = \operatorname{asin}(x) + c$

<u>Oss</u>: $\int \left(k \cdot f(x) + h \cdot g(x)\right) dx$

$= k \cdot \int f(x)\,dx + h \cdot \int g(x)\,dx$