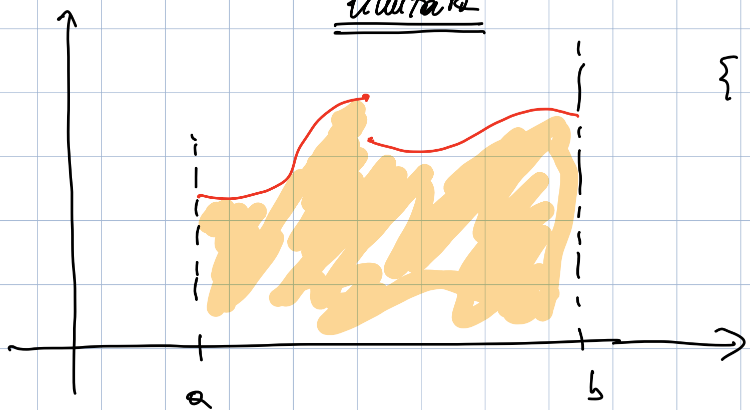
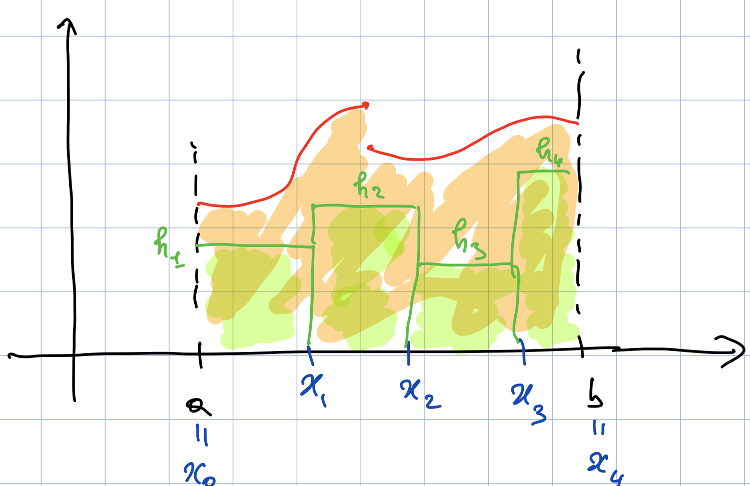
Ist. Mat. I - CIA
15/2/2023

Idea: data $f: [a,b] \to \mathbb{R}$, $f \geq 0$, limitata, calcolare area

$\{(x,y) : x \in [a,b],\ 0 \leq y \leq f(x)\}$



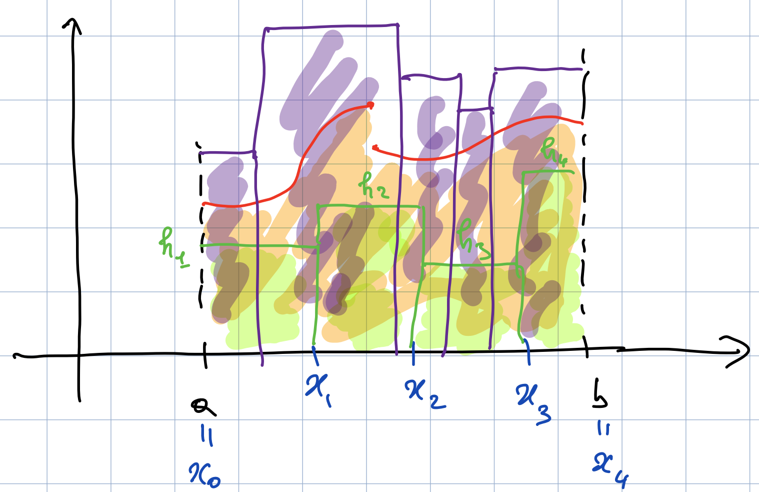
Def: chiamo plurirettangolo con base $[a,b]$ una suddivisione $a = x_0 < x_1 < x_2 < x_3 < \ldots < x_{n-1} < x_n = b$ e una scelta di $h_1, \ldots, h_n \geq 0$; lo indico con P e pongo $A(P) = \sum_{i=1}^{n} h_i \cdot (x_i - x_{i-1})$.



Def: dico che $P \le f$ se $h_i \le f(x)$ $\forall x \in [x_{i-1}, x_i]$

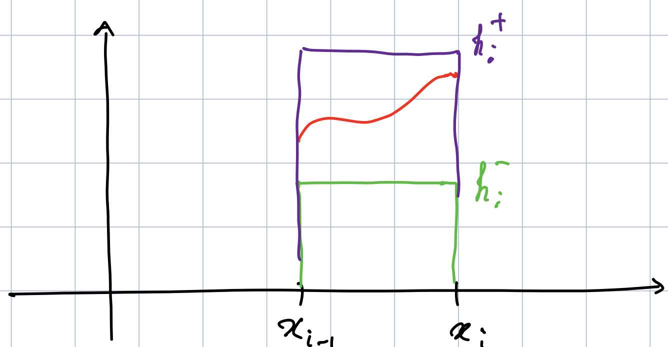
Analogamente $f \le P$.

Lem: se $P_- \le f \le P_+$ allora $A(P_-) \le A(P_+)$



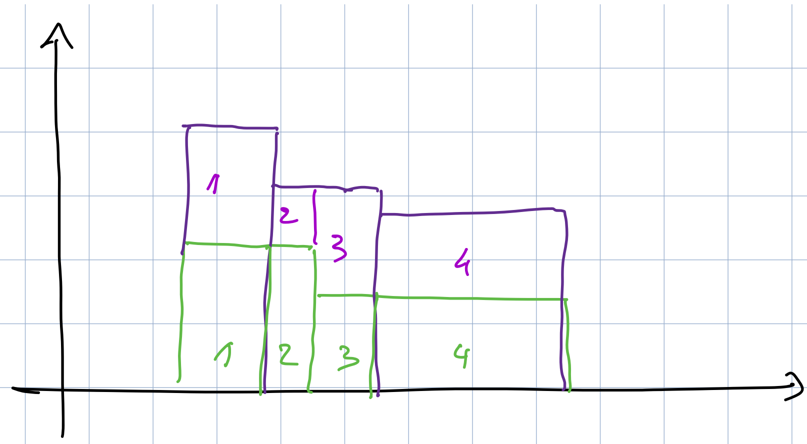
Dimo: chiaro se P_-, P_+ usano la stessa suddivisione

$a = x_0 < x_1 < \dots < x_n = b$



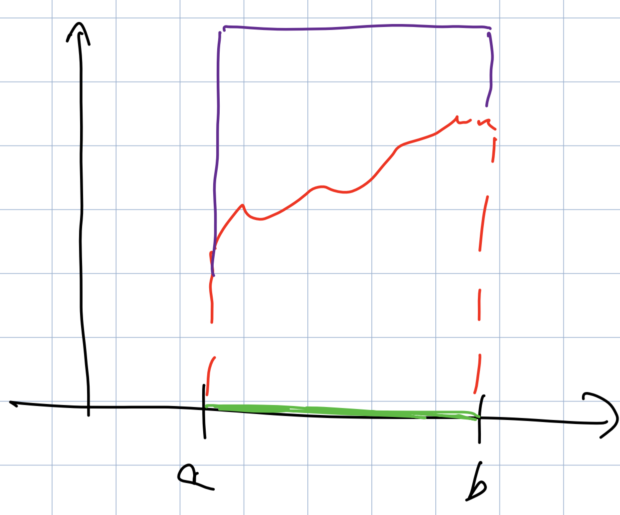
($\le$ vale addendo per addendo)

Fatto: posso sempre usare la stessa suddivisione senza cambiare area:



$\square$

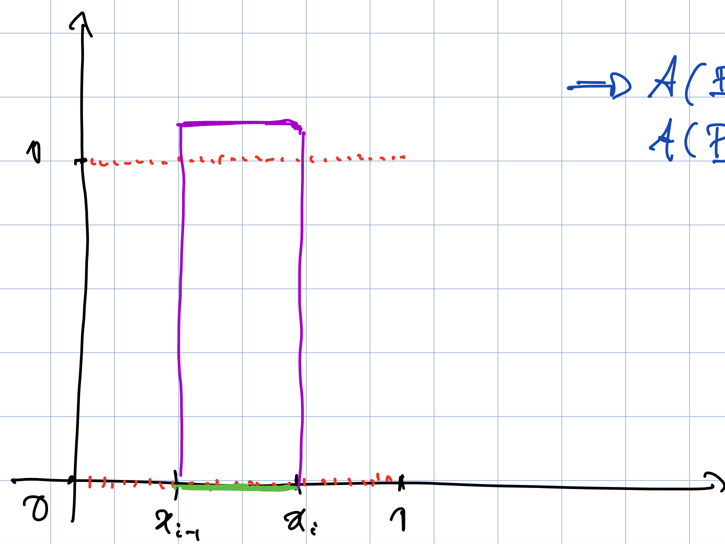
<u>Oss</u>: come P_- posso scegliere $[a,b]\times\{0\}$
se $f(x)\le L \quad \forall x\in[a,b]$
come P_+ posso scegliere $[a,b]\times[0,M]$



<u>Def</u>: se $\sup\{A(P_-) : P_-\le f\}$
$= \inf\{A(P_+) : f\le P_+\}$
dico che f è integrabile su $[a,b]$ e pongo tale valore
$$\int_a^b f(x)\,dx$$

Oss: ci sono funzioni non integrabili

$f:[0,1]\to\mathbb{R}$ $\qquad f(x)=\begin{cases}1 & \text{se } x\in\mathbb{Q}\\ 0 & \text{se } x\notin\mathbb{Q}\end{cases}$



$\Rightarrow A(P_-)=0$

$A(P_+)\geq 1$

Spiego $\int f(x)\,dx$

qualsiasi $\bar{x}_i\in[x_{i-1},x_i]$

$$\underbrace{\sum_{i=1}^{n} h_i^-\cdot(x_i-x_{i-1})}_{\int_a^b f(x)dx} \le \underset{\int_a^b}{\sum_{i=1}^{n}} \underbrace{f(\bar{x}_i)}_{f(x)}\cdot\underbrace{(x_i-x_{i-1})}_{dx} \le \underbrace{\sum_{i=1}^{n} h_i^+\cdot(x_i-x_{i-1})}_{\int_a^b f(x)dx}$$

Oss: $A(P_-)\le A(P_+)\quad \forall\ P_-\le f\le P_+$;

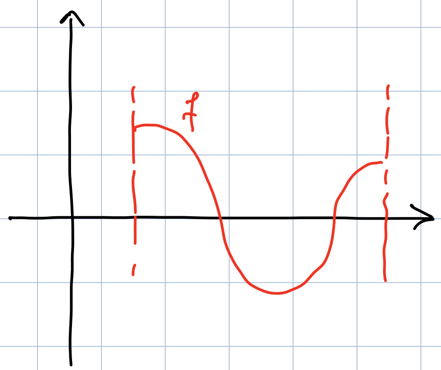
dunque $\exists\int_a^b f(x)\,dx \iff \forall\varepsilon>0\ \exists P_+,P_-$ t.c.

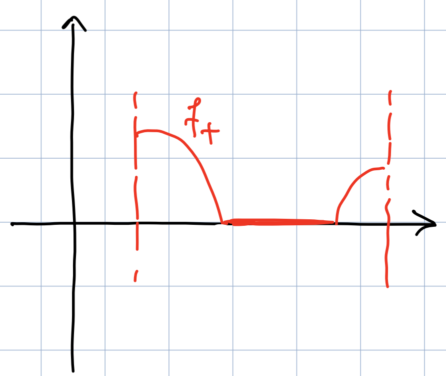
$A(P_+)-A(P_-)<\varepsilon$.

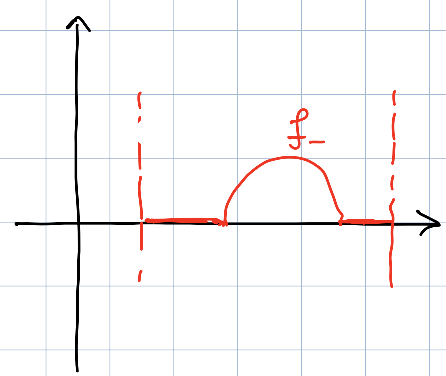
Data $f : [a,b] \to \mathbb{R}$ limitata chiamo

$f_+ \qquad f_+(x) = \max\{f(x), 0\}$

$f_- \qquad f_-(x) = \max\{-f(x), 0\}$



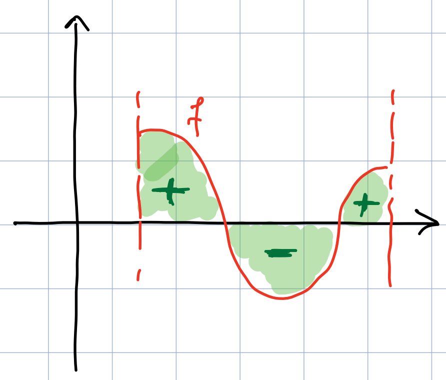


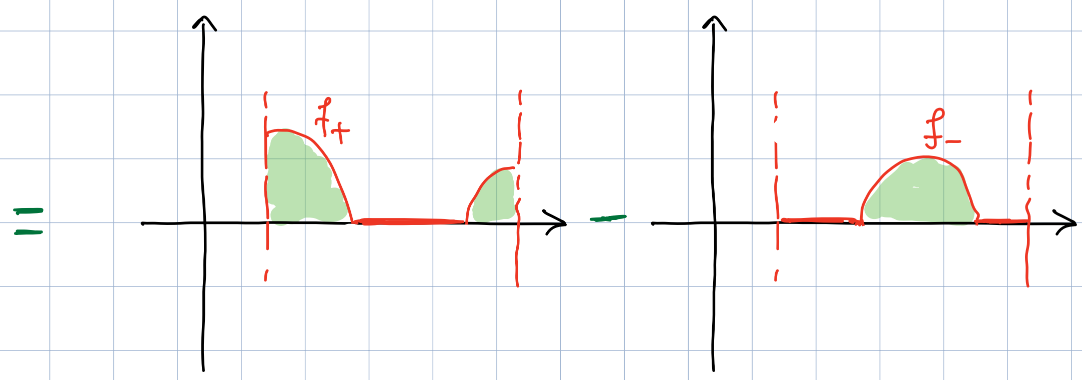


Oss: $f = f_+ - f_-$

Def: dico che f è integrabile se lo sono $f_\pm$ e pongo

$$\int_a^b f(x)\,dx = \int_a^b f_+(x)\,dx - \int_a^b f_-(x)\,dx$$



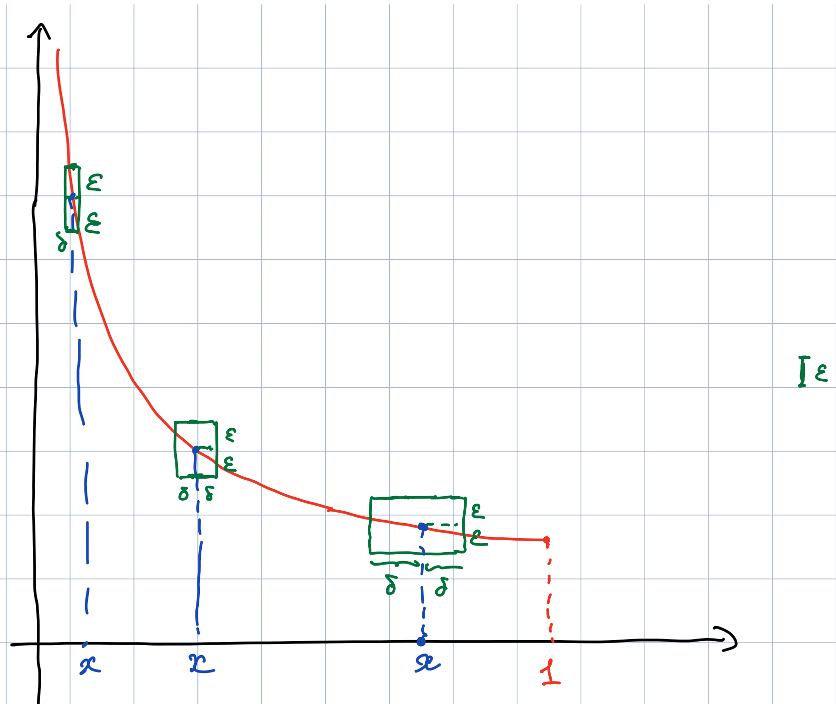


Def: $f: I \to \mathbb{R}$ è continua in $x \in I$ se
$\forall \varepsilon > 0 \quad \exists \delta > 0$ t.c. $|f(y) - f(x)| < \varepsilon$ per $|y-x| < \delta$.

Def: $f: I \to \mathbb{R}$ è uniformemente continua se
$\forall \varepsilon > 0 \quad \exists \delta > 0$ t.c. $|f(y) - f(x)| < \varepsilon$ per $|y-x| < \delta$.

Cioè: dato ε lo stesso δ va bene $\forall x$.

Es: $f(x) = \frac{1}{x}$ su $(0,1]$ è continua
ma non è uniformemente continua:



Teo 1: se $f \geq 0$ è unif. continua su $[a,b]$ esiste $\int_a^b f(x)\,dx$

Teo 2: f continua su $\underbrace{[a,b]}_{\text{chiuso e limitato}}$ $\Longrightarrow$ f uniformemente continua

Teo 3: f continua $\Longrightarrow$ $f_\pm$ continue.

Conseguenze: f continua su $[a,b] \Longrightarrow \exists \int_a^b f(x)\,dx$

f cont $\xRightarrow{3}$ $f_\pm$ cont $\xRightarrow{2}$ $f_\pm$ unif. cont. $\xRightarrow{1}$ $f_\pm$ integrabili
$\Longrightarrow f$ integrabile

Dimo 1: Dato $\varepsilon > 0$ cerco $\Phi_- \leq f \leq \Phi_+$ t.c.
$I(\Phi_+) - I(\Phi_-) < \varepsilon$.

Uso la def. di unif. cont. con $\frac{\varepsilon}{b-a}$.

Trovo $\delta > 0$ t.c. se $|x-y| < \delta$ ho $|f(x)-f(y)| < \varepsilon$.
Suddivido $[a,b]$ in $a = x_0 < x_1 < \dots < x_n = b$ t.c.
$x_i - x_{i-1} < \delta$. Pongo $h_i^- = \min_{[x_{i-1},x_i]} f$, $h_i^+ = \max_{[x_{i-1},x_i]} f$.

$h_i^+ - h_i^- < \varepsilon/(b-a)$.

$$\Rightarrow A(P_+) - A(P_-) = \sum_{i=1}^{n} (h_i^+ - h_i^-)(x_i - x_{i-1})$$

$$\underbrace{\leq \frac{\varepsilon}{b-a} \cdot \underbrace{\sum_{i=1}^{n} (x_i - x_{i-1})}_{b-a}}_{\varepsilon}$$

$\boxtimes$