Ist. Mat. I — CIA

1/12/22

$x \in \mathbb{R}$, $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\left(\displaystyle\sum_{n=0}^{k} \frac{x^n}{n!} + o(x^{k+1}) = e^x \right.$ Taylor ordine k $\left.\right)$

convergente?

$\boxed{x > 0}$ termini positivi $a_n = \dfrac{x^n}{n!}$

rapporto $\dfrac{a_{n+1}}{a_n} = \dfrac{x^{n+1}/(n+1)!}{x^n/n!} = \dfrac{x}{n+1} \longrightarrow 0$

$\Longrightarrow$ convergenza

$\boxed{x < 0}$ So che basta vedere che è assolutamente convergente

$$\sum_{n=0}^{\infty} \left| \frac{x^n}{n!} \right| = \sum_{n=0}^{\infty} \frac{|x|^n}{n!} < +\infty \quad \text{poiché } |a| > 0$$

Q: è vero che ogni serie convergente è assolutamente convergente?

A: NO.

Es: $\sum_{n=1}^{\infty} \frac{1}{n}$ divergente (lo vedremo)

Fatto: $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$ è convergente.

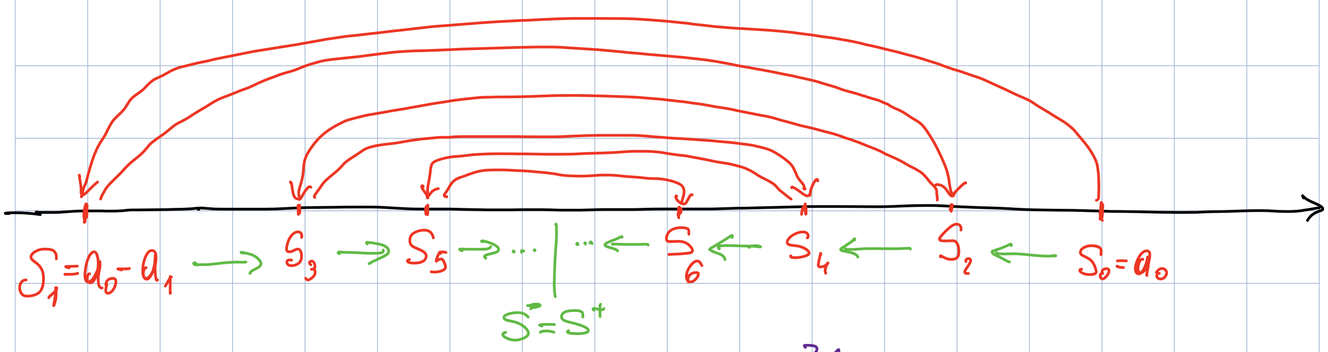
Segue da:

Criterio di Leibniz: data $(a_n)_{n=0}^{\infty}$ con $a_n > 0 \ \forall n$, $a_{n+1} < a_n$, $\boxed{\lim_{n \to \infty} a_n = 0}$

la serie $\sum_{n=0}^{\infty} (-1)^n \cdot a_n$ converge.

Infatti: informalmente:

$$S_m = \sum_{n=0}^{m} (-1)^n \cdot a_n$$ devo vedere che converge.



formalmente: $b_m = S_{2m} = \sum_{n=0}^{2m} (-1)^n a_n$

$c_m = S_{2m+1} = \sum_{n=0}^{2m+1} (-1)^n a_n$

Affermo che $\boxed{b_n \downarrow}$, $\boxed{c_n \nearrow}$, $\boxed{c_n < b_n \quad \forall n}$

1 2 3

$\Rightarrow c_n < b_0 \quad \forall n$

$b_n > c_0 \quad \forall n$

$\Rightarrow$ limitate

$\Rightarrow b_n \to b \ , \ c_n \to c \ ; \ \boxed{b_n - c_n \to 0}$ 4

$\Rightarrow b = c \Rightarrow$ convergente

1
$$b_{n+1} = \sum_{n=0}^{2(n+1)} (-1)^n a_n = \sum_{n=0}^{2n} (-1)^n a_n - a_{2n+1} + a_{2n+2}$$

$$= b_n - \underbrace{(a_{2n+1} - a_{2n+2})}_{>0}$$

$$< b_n$$

2
$$c_{n+1} = \sum_{n=0}^{2(n+1)+1} (-1)^n a_n = \sum_{n=0}^{2n+1} (-1)^n a_n + a_{2n+2} - a_{2n+3}$$

$$= c_n + \underbrace{(a_{2n+2} - a_{2n+3})}_{>0}$$

$$> c_n$$

3
$$c_n = b_n - \underbrace{a_{2n+1}}_{>0} < b_n$$

4
$$b_n - c_n = a_{2n+1} \longrightarrow 0$$

$\boxtimes$

Serie di Taylor:

Sia $f: I \to \mathbb{R}$ derivabile infinite volte; $x_0 \in I$

Sappiamo $\forall n$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + o\left((x-x_0)^{n+1}\right)$$

Q: $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$ converge a $f(x)$?

A: Non sempre.

Esempio: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$

Affermo che $f^{(k)}(0)$ esiste $\forall k$ e vale 0.

Dunque la serie di Taylor di f in $x_0 = 0$ fa identicamente 0: $\displaystyle\sum_{k=0}^{\infty} \frac{0}{k!} \cdot x^k \equiv 0$.

Invece f non è sempre 0.

Certamente $f^{(k)}(x)$ esiste $\forall x \neq 0$; provo che $f^{(k)}(x) \to 0$ per $x \to 0$ $\forall k$.

(Ne segue che $f^{(k)}(0)$ esiste e fa 0.)

$$f(x) = e^{-1/x^2}$$

$$f'(x) = e^{-1/x^2} \cdot (2x^{-3})$$

$$f''(x) = e^{-1/x^2} \cdot (4x^{-6}) + e^{-1/x^2} \cdot (-6x^{-4})$$

$$= e^{-1/x^2} \cdot \frac{4-6x^2}{x^6}$$

Finora succede che $\boxed{f^{(k)}(x) = e^{-1/x^2} \cdot \frac{P_k(x)}{Q_k(x)}}$

P_k, Q_k polinomi.

Provo che se succede ciò per k, succede anche per $k+1$

$\Longrightarrow$ succede sempre (induzione).

Infatti:

$$f^{(k+1)}(x) = e^{-1/x^2} \cdot (-2x^{-3}) \cdot \frac{P_k(x)}{Q_k(x)} + e^{-1/x^2} \cdot \frac{P_k'(x) \cdot Q_k(x) - P_k(x) \cdot Q_k'(x)}{Q_k(x)^2}$$

$$= e^{-1/x^2} \cdot \frac{(-2) \cdot P_k(x) \cdot Q_k(x) + x^3 (P_k' \cdot \dots \quad)}{x^3 \cdot Q_k(x)^2}$$

$$= e^{-1/x^2} \cdot \frac{P_{k+1}(x)}{Q_{k+1}(x)}$$

<u>Ora</u>: $e^{-1/x^2} \cdot \frac{P_k(x)}{Q_k(x)} \longrightarrow 0 \qquad \forall k$

Q: la serie di Taylor di f infinitamente derivabile converge a f?

A: non sempre ma per le f "solite" sì

Es: $f(x) = e^x$. Fissato x, $\forall n$ esiste c tra 0 e x (Taylor/Lagrange)

$$f(x) = \sum_{k=0}^{n} \frac{x^k}{k!} + \underbrace{\frac{e^c}{(n+1)!} \cdot x^{n+1}}$$

c dipende da n ma è sempre compreso tra 0 e x
$\Rightarrow$ resto limitato

$$\frac{e^c}{(n+1)!} \cdot x^{n+1} \longrightarrow 0$$

infatti $\frac{x^{n+1}}{(n+1)!} \longrightarrow 0$ per criterio rapporto

Conclusione $\quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Questo argomento non funziona per $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$

perché $f^{(k)}(c)$ non rimane limitato per c tra 0 e x.

Analog.

$$\boxed{\begin{aligned} \cos(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \\ \sin(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \\ \log(1+x) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot x^{n+1} \\ (1+x)^{\alpha} &= \sum_{n=0}^{\infty} \binom{n}{\alpha} \cdot x^n \end{aligned}}$$

Zanichelli: pag. 170

(28)

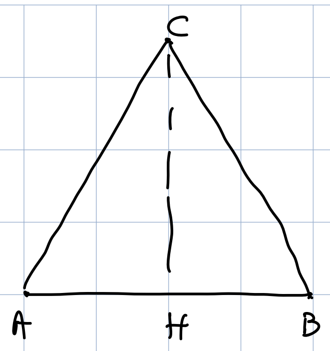
$$V(r) = \frac{4}{3}\pi r^3 \qquad V'(r) = 4\pi r^2$$

$$S(r) = 4\pi r^2$$

$$W(S) = \frac{4}{3}\pi \left(\sqrt{\frac{S}{4\pi}} \right)^3 = \frac{1}{3\sqrt{\pi}} \cdot S^{3/2}$$

$$W'(S) = \frac{1}{2\sqrt{\pi}} \cdot S^{1/2}$$

(29)



$\overline{AB} = 3\,cm$

area cresce a $4\,\frac{cm^2}{s}$

come cresce altezza?

come cresce $\overline{BC}$?

$\overline{CH} = h(t)\,cm$

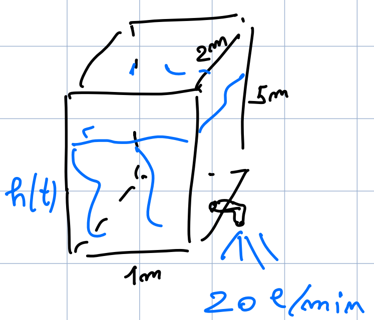
$A(t) = \frac{3}{2}\, h(t) \cdot cm^2$

velocità crescita di A è $A'(t) = \frac{3}{2}\, h'(t) \cdot \frac{cm^2}{s}$

$\Rightarrow h'(t) = \frac{8}{3} \Rightarrow h(t) = \frac{8}{3}t + h_0$

$\overline{BC} = \sqrt{(3/2)^2 + h(t)^2} \quad : \quad \frac{d\,BC}{dt} = \dots$

(30)



$V(t) = 1 \cdot 2 \cdot h(t) \cdot m^3$

$V'(t) = 2h'(t) \cdot \frac{m^3}{min} = -20\,\frac{l}{min}$

$= -\frac{20}{1000} \cdot \frac{m^3}{min}$

$h'(t) = -\frac{1}{100}$

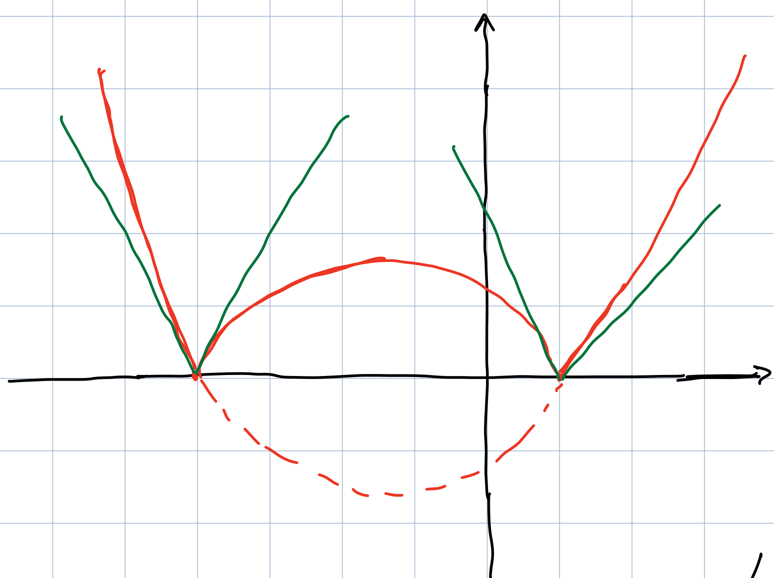
$\Rightarrow$ h cala di 1 cm/min

Studiare f vicino ai punti in cui f' potrebbe non esistere

(Sempre: intendiamo $f: D \to \mathbb{R}$ dove D è il più grande su cui $f(x)$ ha senso)

32) $|x^2+3x-4|$

$$|x^2+3x-4| = |(x+4)(x-1)|$$



$$f'_-(-4) = (2x+3)\Big|_{x=-4} = -5$$

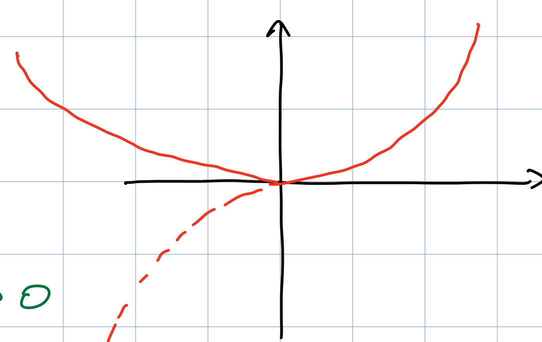
$$f'_+(-4) = +5$$

$$f'_-(1) = -5$$

$$f'_+(1) = 5$$

due cuspidi

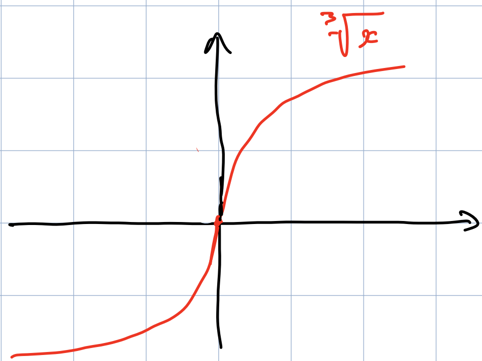
33) $$|x^3| = \begin{cases} x^3 & x \geq 0 \\ -x^3 & x < 0 \end{cases}$$



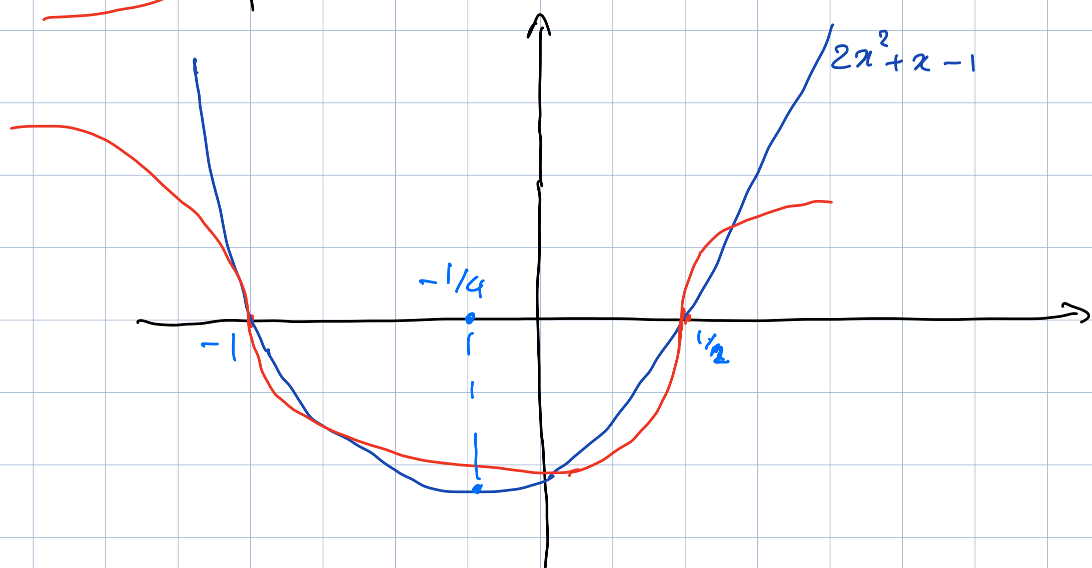
$$f'(x) = \begin{cases} 3x^2 & x \geq 0 \\ -3x^2 & x < 0 \end{cases} \xrightarrow[x \to 0]{} 0$$

Analog. $f''(0) = 0$ mentre $f'''(0)$ non esiste

(34) $\sqrt[3]{2x^2+x-1}$ $\qquad f(x) = (2x^2+x-1)^{1/3} \qquad f: \mathbb{R} \to \mathbb{R}$



$$\sqrt[3]{2x^2+x-1} = \sqrt[3]{(2x-1)(x+1)}$$



Facendo calcoli: $\qquad f'_{\pm}(1/2) = +\infty$

$$f'_{\pm}(-1) = -\infty$$

$$f'(x) = \underbrace{\frac{1}{3}\,\frac{1}{(\dots)^{2/3}}}_{>0} \cdot \underbrace{(4x+1)}_{\text{concorde con } x+1/4}$$

min in $x = -1/4$

$$f''(x) = -\frac{2}{9} \cdot \frac{1}{(\dots)^{+5/3}} \cdot (4x+1)^2$$

$$+ \frac{1}{9} \cdot \frac{1}{(\dots)^{2/3}} \cdot 4$$

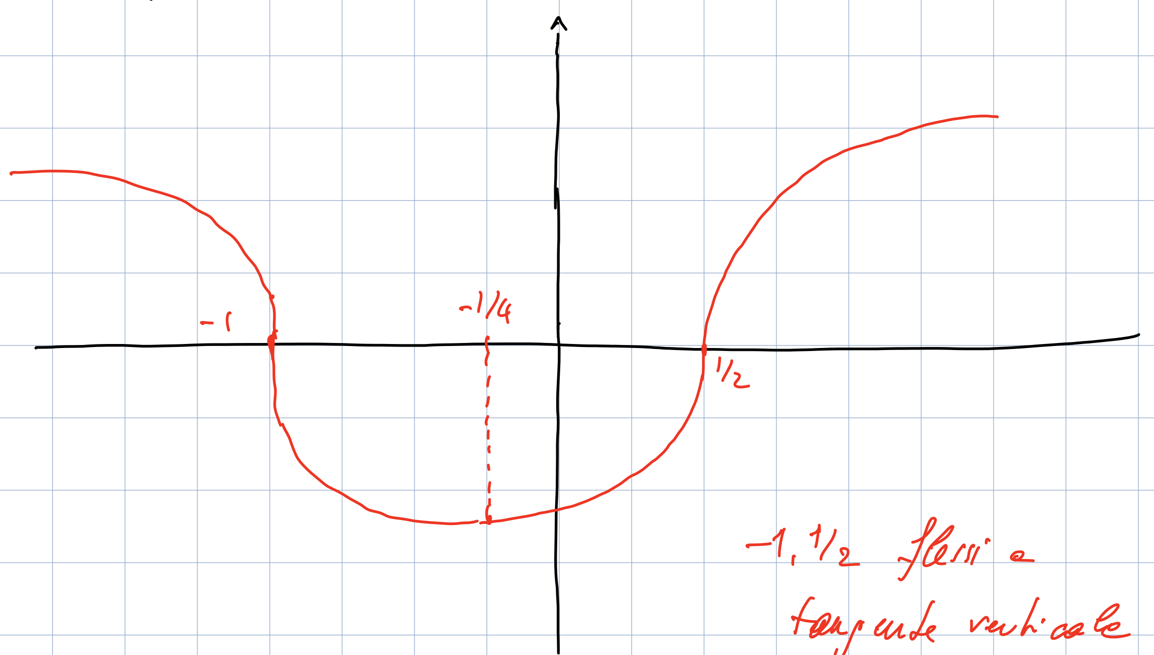
$$= \frac{1}{9} \frac{1}{(\dots)^{5/3}} \cdot \left(-2(4x+1)^2 + 12\,(2x^2+x-1)\right)$$

$$= \frac{1}{9} \cdot \frac{1}{(\dots)^{5/3}} \cdot \left(\begin{array}{l} -32x^2 - 16x - 2 \\ +24x^2 + 12x - 12 \end{array} \right)$$

$$= -\frac{2}{9} (2x^2+x-1)^{-5/3} \cdot (\underbrace{6x^2+2x+7}_{\Delta < 0})$$

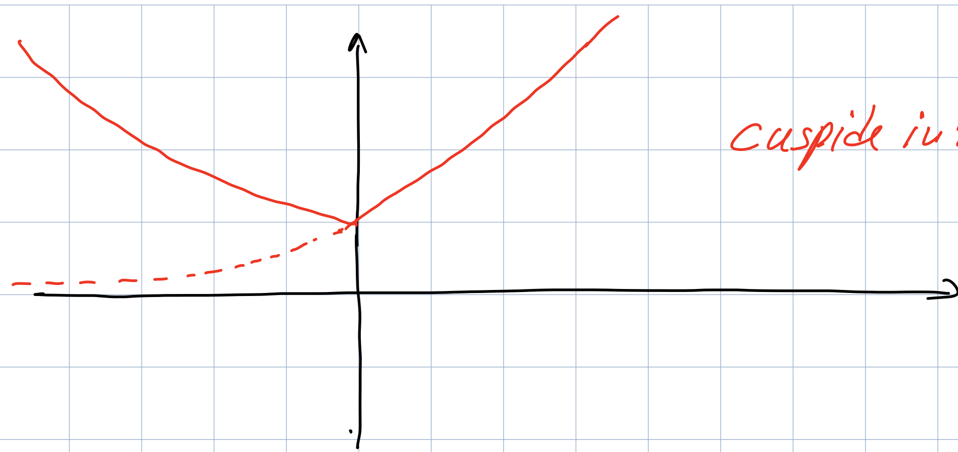
$\Rightarrow$ sempre positivo

$\Rightarrow f''(x)$ discorde da $2x^2+x-1$:



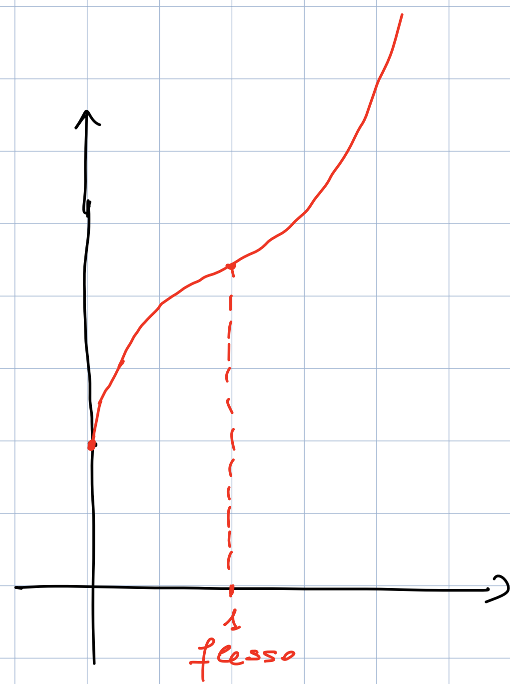
$-1, \ 1/2$ flessi a tangente verticale

(35) $e^{-|x|}$



cuspide in $x=0$

(36) $e^{\sqrt{x}}$ $\quad [0,+\infty) \to \mathbb{R}$



$$f'(x) = e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$$

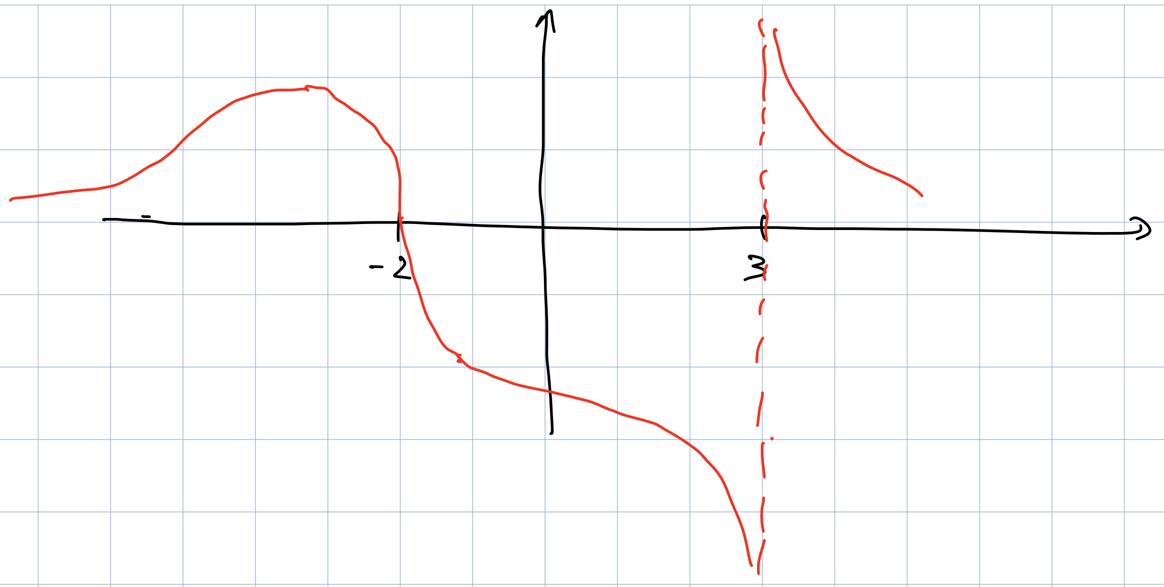
$$f''(x) = e^{\sqrt{x}} \cdot \frac{1}{4x} + e^{\sqrt{x}} \cdot \frac{1}{2} \cdot \left(-\frac{1}{2}\right) \cdot x^{-3/2}$$

$$= e^{\sqrt{x}} \cdot \frac{1}{4x}\left(1 - \frac{1}{\sqrt{x}}\right)$$

(37) $e^{x} \cdot \sqrt[3]{\frac{x+2}{x-3}}$ $\quad \mathbb{R} \to \mathbb{R}$

vicino a $x = -2$ si comporta come $-c \cdot (x+2)^{1/3}$

vicino a $x = 3$ si comporta come $c \cdot (x-3)^{-1/3}$



-2
3