Ist. Mat. I – CIA

28/9/22

<u>Prop</u>: $\forall n \in \mathbb{N}$ $\quad \sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

<u>Dim</u>: induzione.

$n = 0$ (P. B) $\quad \underset{\substack{\| \\ 0}}{\sum_{k=0}^{0} k^2} = \underset{\substack{\| \\ 0}}{\frac{0(0+1)(2\cdot 0+1)}{6}}$ ✓

P.I. Suppongo $\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Devo vedere: $\sum_{k=0}^{n+1} k^2 = \boxed{\frac{(n+1)(n+2)(2n+3)}{6}}$.

$=?$

$$\boxed{\sum_{k=0}^{n+1} k^2} = \sum_{k=0}^{n} k^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= \frac{n+1}{6}\left(2n^2+n+6n+6\right) = \boxed{\frac{n+1}{6}\left(2n^2+7n+6\right)}$$

$$(n+2)(2n+3) = 2n^2+3n+4n+6 = 2n^2+7n+6.$$

▨

Foglio 1 - Ese 1

(a) $\{1, 2, 3, 6\}$

(b) $\{n < 8 :$ $|n-8|$ si scrive con 3 lettere $\}$

$\cup \; \{n \geq 8 :$ n si scrive con $n-8$ si scrive con 8 lettere $\}$

$= \{2, 5, 6, 7\} \cup \{23, 26, 30, 31, 34, 39, 109, 110, 48, 68, 111, 114 \ldots\}$

(d) $\{x \in \mathbb{Q} : 4x^2 - 1 = 0\} = \{x = \pm \frac{1}{2}\}$

(e) $\{x \in \mathbb{Z} : 4x^2 - 1 = 0\} = \emptyset$

(f) $\{x \in \mathbb{Q} : x^2 + 1 < 0\} = \emptyset$

(g) $\{z, u, d, t, q, c, s, o, n, v, m\}$

(h) $\{(i, j) \in \mathbb{N} \times \mathbb{N} : \ldots\}$

MATEMATICA

$\{(1,1), (2,2), \ldots, (10,10), (1,5), (5,1), (2,6), (6,2), (2,10), (10,2), (6,10), (10,6), (3,7), (7,3)\}$

Ese 2:

(a) $X \subsetneq Y$ $\quad [X \subseteq Y, \; Y \not\subseteq X]$

(b) nessuna delle due

(c) $X = Y$ $[X \subseteq Y, \quad Y \subseteq X]$

(d) $Y \subsetneq X$

Convenzione

$X \subseteq Y$ vera anche se $X = Y$

Non uso $\subset$

(e) $Y \subsetneq X$

(f) $X = \{x \in \mathbb{N} : x \text{ primo}\}$
$Y = \{y \in \mathbb{N} : y \text{ non è multiplo di } 5\}$

$X \nsubseteq Y$ No: $5 \in X$, $5 \notin Y$

$Y \nsubseteq X$ No: $6 \in Y$, $6 \notin X$

(g) $X = Y$
(h) $X \subsetneq Y$

<u>Ese 3</u>: (a) $23 : 5$ $23 = 4 \cdot 5 + 3$

(b) $162 : 7$ $162 = 23 \cdot 7 + 1$

(c) $78 : 17$ $78 = 4 \cdot 17 + 10$

(d) $133 : 31$ $133 = 4 \cdot 31 + 9$

Ese 4

$$\sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha} \qquad |\alpha| < 1$$

somma della serie geometrica di ragione α

(a) $0,\overline{4} = \sum_{k=1}^{\infty} 4 \cdot 10^{-k} = 4 \cdot \frac{1}{10} \cdot \sum_{k=0}^{\infty} (10^{-1})^k$

$$4 \cdot \frac{1}{10} \cdot \left(10 \cdot \sum_{k=1}^{\infty} 10^{-k}\right)$$

$$= 4 \cdot \frac{1}{10} \cdot \frac{1}{1-\frac{1}{10}} = \frac{4}{9}$$

(b) $5,\overline{4} = 5 + 0,\overline{4} = 5 + \frac{4}{9} = \frac{5 \cdot 9 + 4}{9} = \frac{49}{9}$

$$\| $$

$$\frac{5 \cdot (10-1) + 4}{9} = \frac{5 \cdot 10 + 4 - 5}{9} = \frac{54-5}{9}$$

(c) $0,2\overline{3} = 2 \cdot 10^{-1} + \sum_{k=2}^{\infty} 3 \cdot 10^{-k}$

$$= \frac{2}{10} + 3 \cdot \frac{1}{100} \cdot \sum_{k=0}^{\infty} (10^{-1})^k$$

$$= \frac{2}{10} + \frac{3}{100} \cdot \frac{1}{1-\frac{1}{10}} = \frac{2}{10} + \frac{3}{90} = \frac{18+3}{90} = \frac{21}{90} = \frac{7}{30}$$

$$\|$$

$$\frac{2\cdot 9+3}{90} = \frac{2\cdot(10-1)+3}{90} = \frac{2\cdot 10+3-2}{90}$$

$$= \frac{23-2}{90}$$

(d) $0,\overline{37} = \underbrace{3\cdot 10^{-1} + 7\cdot 10^{-2}} + 3\cdot 10^{-3} + 7\cdot 10^{-4} + 3\cdot 10^{-5} + \dots$

$$= 37\cdot 10^{-2} + 37\cdot 10^{-4} + 37\cdot 10^{-6} + \dots$$

$$= 37\cdot \sum_{k=1}^{\infty} 10^{-2k}$$

$$= 37\cdot \frac{1}{100}\cdot \sum_{k=0}^{\infty} \left(10^{-2}\right)^{k}$$

$$= \frac{37}{100}\cdot \frac{1}{1-\frac{1}{100}} = \frac{37}{99}$$

Esercizio: verificare la regola in generale

$$n,a\overline{p} = \frac{nap - na}{\underbrace{9\dots}_{\text{lungh. } p}\,\underbrace{9\,0\dots0}_{\text{lungh. } a}}$$

$f : X \to Y$ iniettiva se $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

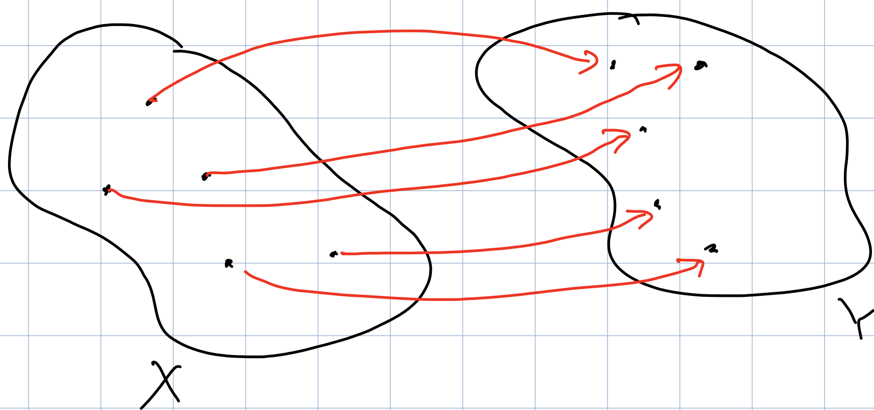
suriettiva se $\forall y \in Y \ \exists x$ t.c. $f(x) = y$

Def: $\{ f(x) : x \in X \} \subseteq Y$ Immagine di f

$\mathrm{Im}(f)$

f suriettiva se $\mathrm{Im}(f) = Y$.

Def: f bijettiva se è iniettiva e suriettiva.



Composizione: date $f : X \to Y$, $g : Y \to Z$ ho

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

nuova funzione "composta" $g \circ f : X \to Z$

$(g \circ f)(x) = g(f(x))$

Es: $f: \mathbb{N} \to \mathbb{N}$ $\quad f(n) = n^2$
$g: \mathbb{N} \to \mathbb{N}$ $\quad g(n) = 2n$

$g \circ f$: $\begin{tikzcd} \mathbb{N} \arrow[r, "f"] \arrow[rr, bend right] & \mathbb{N} \arrow[r, "g"] & \mathbb{N} \end{tikzcd}$

$$(g \circ f)(n) = g(f(n)) = g(n^2) = 2n^2$$

$f \circ g$: $\begin{tikzcd} \mathbb{N} \arrow[r, "g"] \arrow[rr, bend right] & \mathbb{N} \arrow[r, "f"] & \mathbb{N} \end{tikzcd}$

$$(f \circ g)(n) = f(g(n)) = f(2n) = 4n^2$$

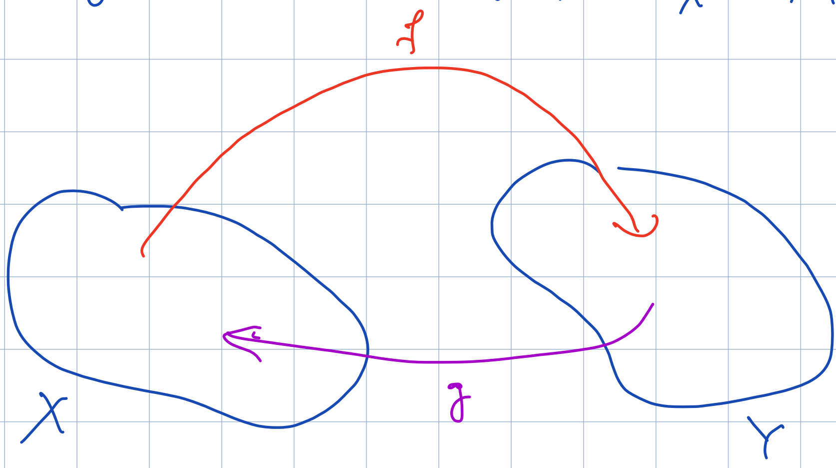
Due funzioni sono uguali se hanno stesso dominio, stesso codominio, e lo stesso valore su ogni pto del dominio.

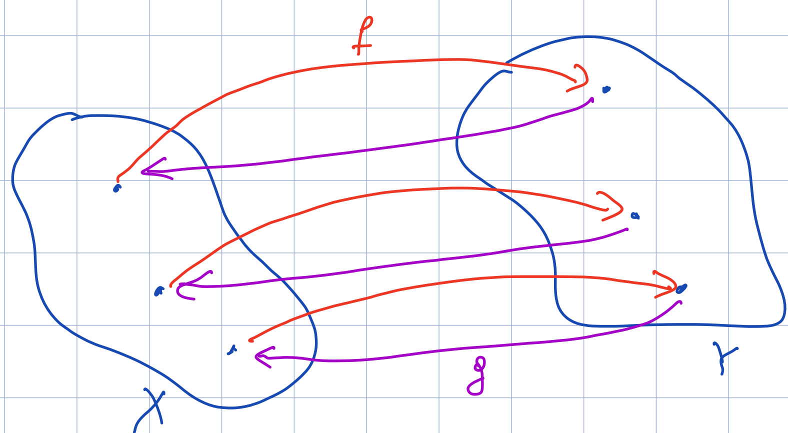
Nell'esempio $g \circ f \neq f \circ g$ perché
$(g \circ f)(3) = 18$
$(f \circ g)(3) = 36$

$f: \mathbb{N} \to \mathbb{Z}$ $\quad f(n) = -1 \quad \forall n$
$g: \mathbb{N} \to \mathbb{Z}$ $\quad g(n) = \cos((2n+1) \cdot \pi) \quad \forall n$
$f = g$

Funzione identità: $id_X : X \to X$
$x \mapsto x$

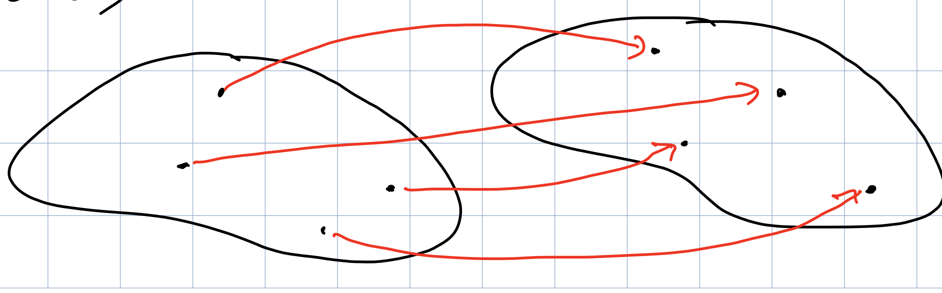
Def: $f: X \to Y$ è detta invertibile se esiste
$g: Y \to X$ t.c. $g \circ f = id_X$, $f \circ g = id_Y$





Tale g se esiste è detta inversa di f e indicata f^{-1}.

Prop: f è invertibile se e solo se è bigettiva
(Esercizio)



inversa di biiettiva ottenuta invertendo le frecce.

Es: $f: \mathbb{N} \to \mathbb{Z}$ $\quad f(n) = (-1)^n \cdot \left[\frac{n+1}{2}\right]$

$[x]$ = "parte intera di $x \in \mathbb{R}$"

$$f: \begin{array}{cccccccccc} 0, & 1, & 2, & 3, & 4, & 5, & 6, & 7, & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 0 & -1 & 1 & -2 & 2 & -3 & 3 & -4 & \end{array}$$

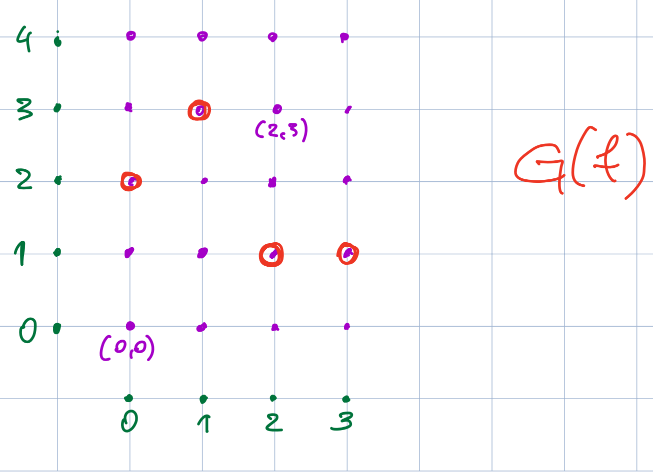
Def: data $f: X \to Y$ chiamo grafico di f

$$G(f) = \{ (x, f(x)) \in X \times Y \ : \ x \in X \}$$

Es: $X = \{0, 1, 2, 3\}$

$Y = \{0, 1, 2, 3, 4\}$

$f: X \to Y \quad f(x) = $ resto di $(x^2 + 2) : 5$



Operazioni sugli insiemi numerici. Fatto: su $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ ho due operazioni binarie interne $+, \cdot$

$$+ : X \times X \longrightarrow X \qquad\qquad \cdot : X \times X \longrightarrow X$$
$$(x_1, x_2) \longmapsto x_1 + x_2 \qquad\qquad (x_1, x_2) \longmapsto x_1 \cdot x_2$$

X = uno qualsiasi di $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

Proprietà delle operazioni a seconda di X

	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$
1. $\exists 0 \in X$ t.c. $0 + x = x + 0 = x$; 0 el. neutro +	✓	✓	✓	✓
2. $x + (y+z) = (x+y) + z \quad \forall x,y,z$; associativa +	✓	✓	✓	✓
3. $x + y = y + x \quad \forall x, y$; commutativa +	✓	✓	✓	✓
4. $\forall x \ \exists (-x)$ t.c. $x + (-x) = 0$; esistenza opposto +	✗	✓	✓	✓
5. $\exists 1 \in X$ t.c. $1 \cdot x = x \cdot 1 = x$; 1 el. neutro $\cdot$	✓	✓	✓	✓
6. $x \cdot (y \cdot z) = (x \cdot y) \cdot z \quad \forall x,y,z$; associativa $\cdot$	✓	✓	✓	✓
7. $x \cdot y = y \cdot x \quad \forall x,y$; commut. $\cdot$	✓	✓	✓	✓
8. $\forall x \neq 0 \ \exists x^{-1}$ t.c. $x \cdot x^{-1} = 1$; esistenza inverso $\cdot$	✗	✗	✓	✓
9. $x \cdot (y+z) = x \cdot y + x \cdot z$; distributiva	✓	✓	✓	✓

$(x \cdot y) + (x \cdot z)$

$\mathbb{Q}, \mathbb{R}$ sono <u>campi</u>

Ordinamento su $\mathbb{R}$ $\quad x < y$

Proprietà

- transitiva: $x < y \,,\, y < z \Longrightarrow x < z$
- tricotomia: vale una e una sola tra $x<y, x=y, y<x$
- monotonia: $x < y \Longrightarrow x+z < y+z \quad \forall z$

  $x<y, z>0 \Longrightarrow x \cdot z < y \cdot z$

$y < x$ si scrive $x > y$.

ordinamento largo: $x \leq y$ significa $x<y$ o $x=y$

Proprietà di $x \leq y$

- transitiva ok
- monotonia ok
- riflessiva: $x \leq x$ vera
- antisimmetrica $x \leq y \,,\, y \leq x \Longrightarrow x = y$.