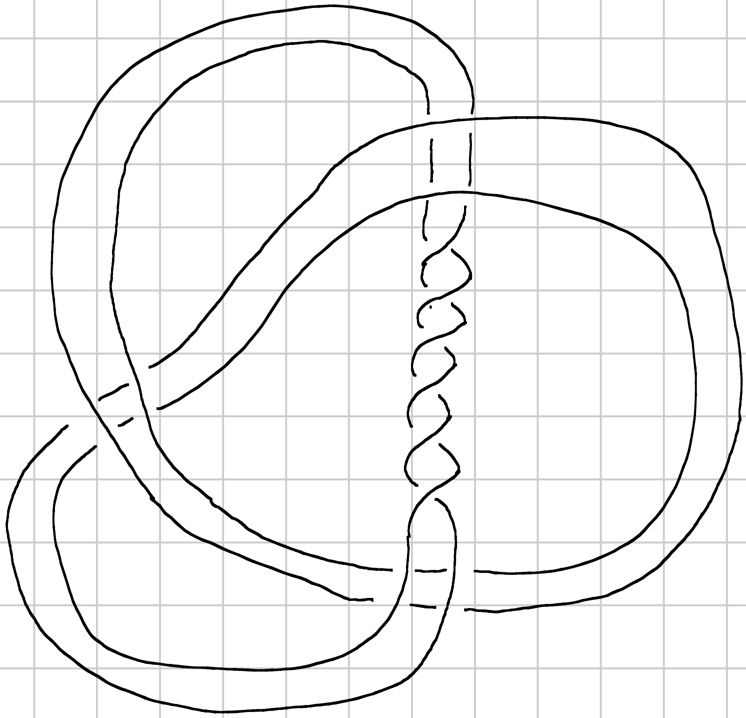
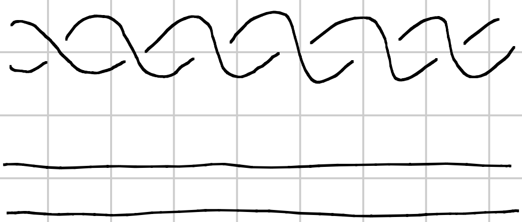
A boundary link which is not split.

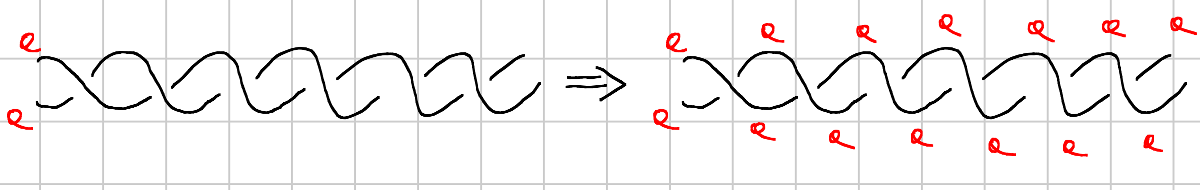


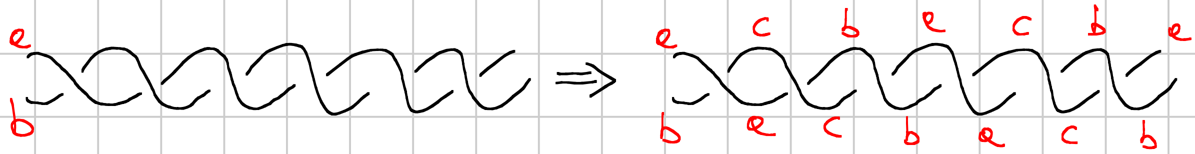
We study 3-colorings of the above link to show it is not split (we already know it is boundary, since it is $K \cup K'$, where K' is a longitude of K).

1. By replacing
   with

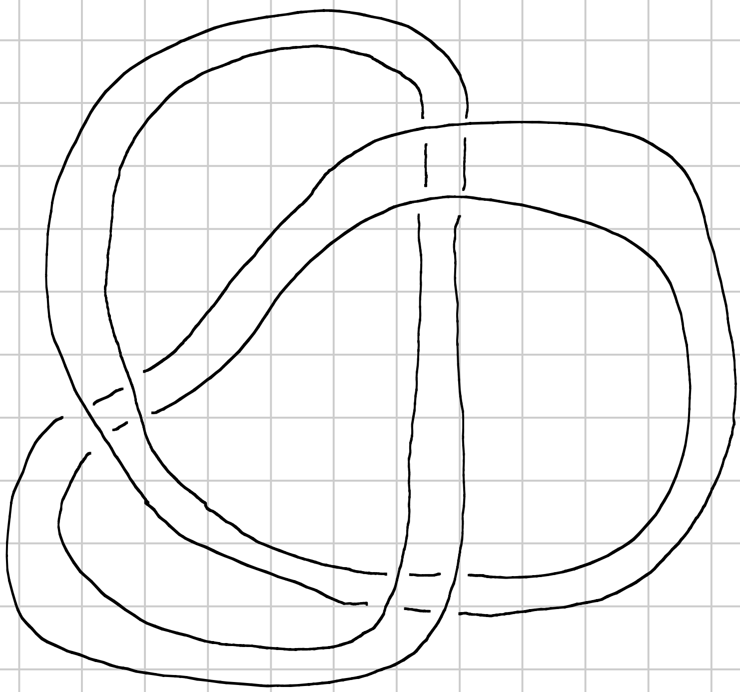


we do not alter colorings. In fact,





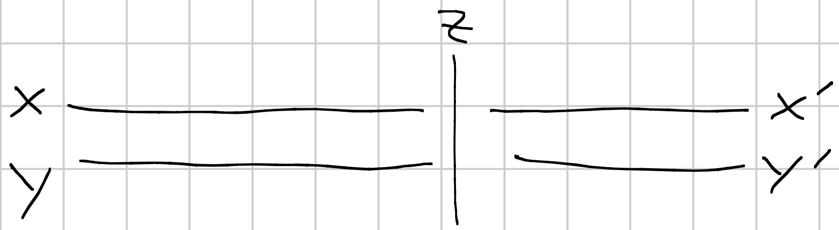
② Therefore, we study colorings of



Recall that we must associate a color to each overarc.

③ Suppose first that two parallel overarcs have the same color. We prove that in this case the coloring is trivial (i.e. constant).

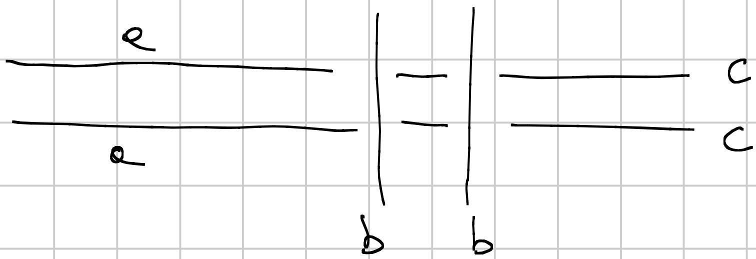
First observe that every pair of parallel arcs have the same color: for any choice of colors x, y, z,



$$x = y \implies x' = y',$$

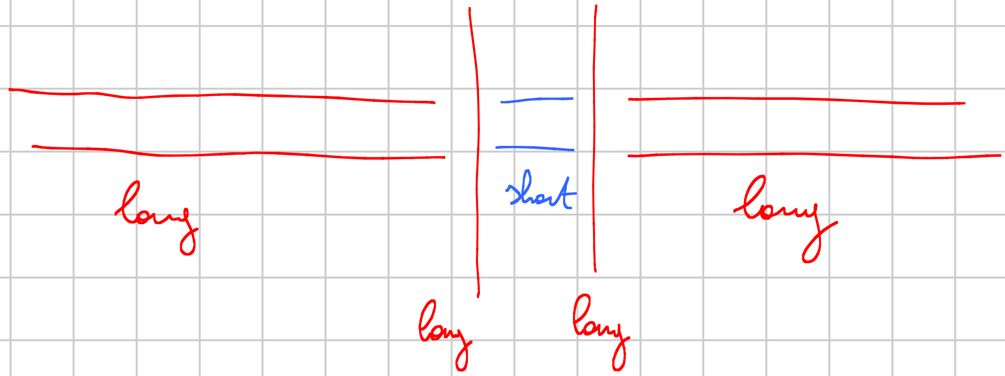
hence the property of having the same colour propagates from a pair of parallel arcs to the adjacent pairs, hence to all the pairs.

Therefore, at any "micro-crossing" (i.e. at any group of 4 crossings as below) the situation is as follows

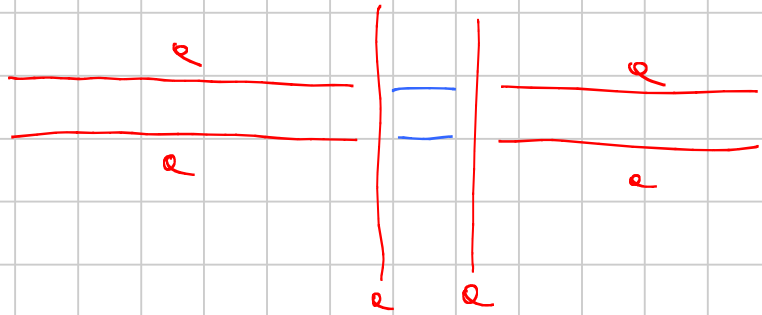


This gives $a = c$ (check the case $a = b$ and $a \neq b$ separately).

Let us call "long arcs" the arcs of the lines corresponding to parallel copies of the arcs of the trefoil, "short arcs" the other ones.



We have proved that, if one pair of parallel arcs share the same color a, then also the adjacent pairs of long parallel arcs have colour a. This implies that all long arcs have colour a.



Finally, this implies that short arcs too have colour a, i.e. the colouring is constant, as claimed.

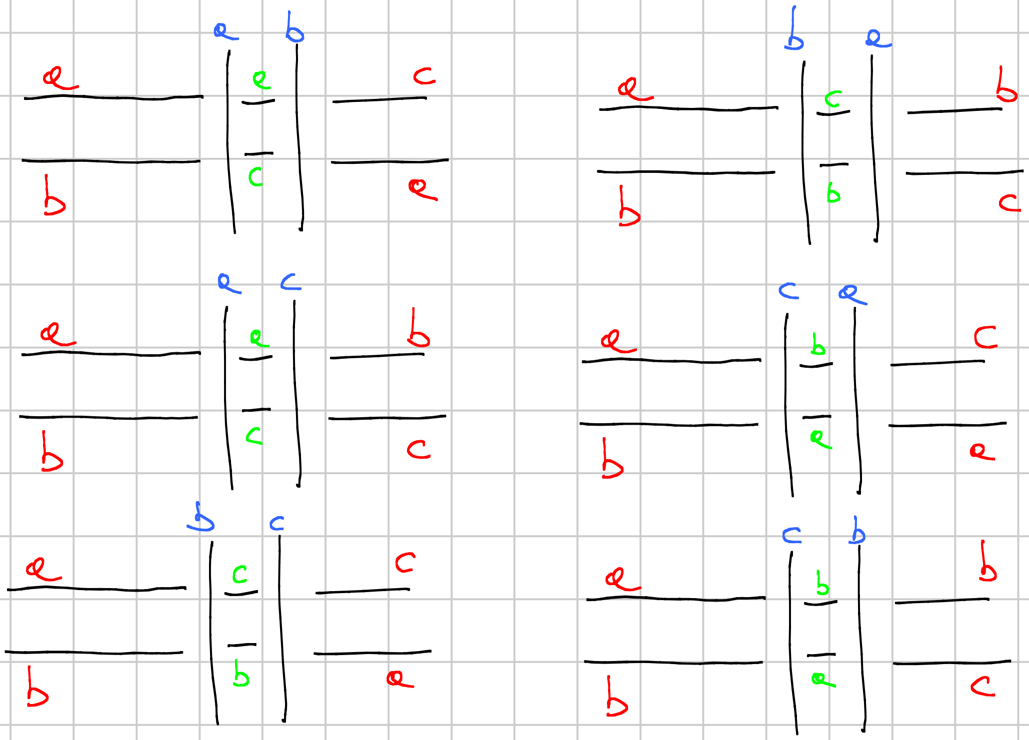
(5) Fix now a pair of parallel arcs, and fix their colors a, b. We will show that this choice extends to at most one global coloring

of the diagram. We have already settled the case $a=b$, hence we suppose $a \neq b$. Suppose that a is the color of the arc lying on K, b the color of the arc lying on K'.

In general, if (e,e') is a pair of long parallel arcs with $e \subseteq K$, $e' \subseteq K'$, we say that the pair has colors (x,y) if x is the color of e, y the color of e'.

Fact 1: If one pair of long parallel arcs has colors (a,b), (b,c) or (c,a) (where a,b,c are pairwise distinct), then every pair of long parallel arcs has colors (a,b), (b,c) or (c,a).

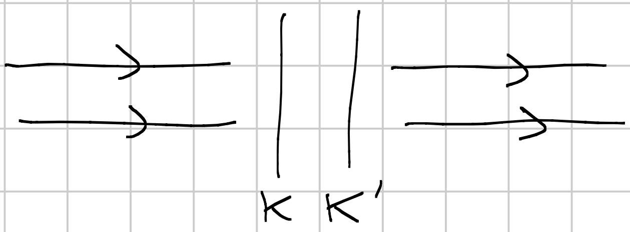
The proof of this fact is based on a case by case analysis. For example, if we start with (a,b) and we pass under a pair of the form (x, y), then:



(the case $x = y$ was ruled out in ③).

These diagrams in fact suffice to prove Fact 1.

Observe we can orient the link so that, at every macrocrossing, the under-arcs pass first under K, then under K'



Fact 2: If a pair of long arcs is labelled by the colors (a, b), then

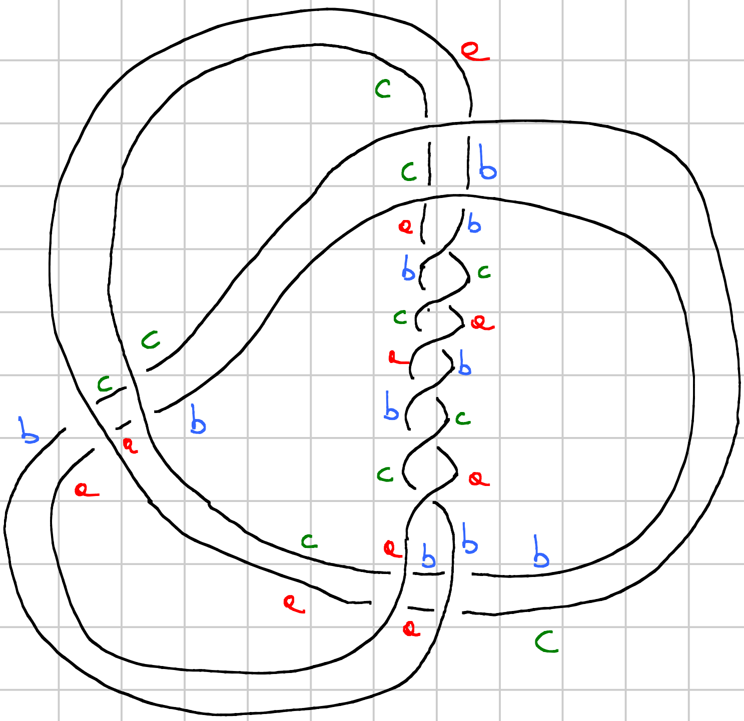
the following pair is labelled by (c, a).
In the same way, (b, c) is followed by (a, b), and (c, a) by (b, c).
Let us prove the first statement, the other ones being completely analogous. By Fact 1, only the following 3 cases are possible:

and this proves Fact 2.
By Fact 2, the choice of a pair of colors for a pair of parallel long arcs completely determines the coloring of all long arcs, hence the coloring of the whole diagram. This proves (4).

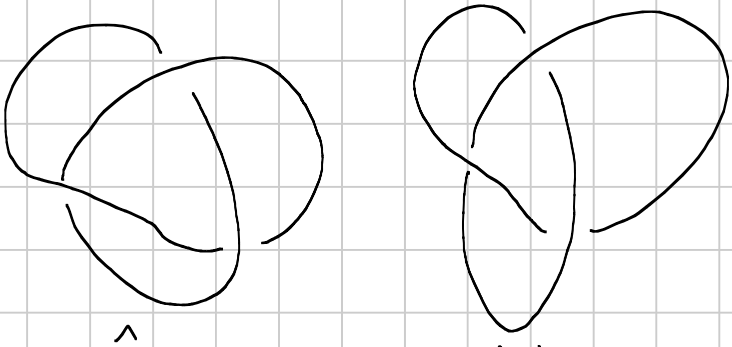
Corollary The diagram of $K \cup K'$ admits at most 9 colorings.

Proof: There are 9 choices for the colors of a pair of parallel long arcs. The conclusion follows from (4).

Remark: In fact, $K \cup K'$ admits exactly 9 colorings. Here is a non-trivial one:



Suppose now by contradiction that $K \cup K' = L$ is split. Since both K and K' are trefoils, L is isotopic to the link $\hat{L} = \hat{K} \cup \hat{K}'$



$\hat{K}$ $\hat{K}'$

We know that $\hat{K}$ admits 9 3-colorings, and the same holds true for $\hat{K}'$. Therefore, $\hat{L}$ admits $9 \cdot 9 = 81$ colorings. Thus L is NOT isotopic to $\hat{L}$, hence L is not split.