Teoria dei Nodi
23/5/2019

Surgery presentation of 3-manifolds.

Dehn filling

$T \cong S^1 \times S^1 \qquad H_1(T) = \mathbb{Z} \times \mathbb{Z} = \langle \alpha, \beta \mid [\alpha, \beta] \rangle$

$$\left\{ \begin{array}{l}\text{simple closed curves} \\ \text{on } T \text{ not bounding disc}\end{array} \right\} \Big/ \text{isotopy} \longleftrightarrow \left\{ \underset{\text{coprime}}{(p,q) \in \mathbb{Z} \times \mathbb{Z}} \right\} \Big/ \pm 1 \longleftrightarrow \mathbb{Q} \cup \{\infty\}$$

$$\gamma \longleftrightarrow [\gamma] = \pm(p\alpha + q\beta) \longleftrightarrow p/q$$

$M^{(3)}$ manifold ; $\partial M \supset T$, $\gamma \subset T$ s.c.c. non-trivial

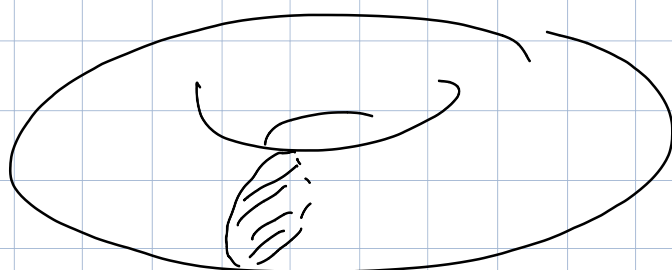
$$M(\gamma) = M \cup_f \underbrace{(D^2 \times S^1)}_{\text{solid torus}} \qquad \begin{array}{l} f : \partial D^2 \times S^1 \to T \\ f(\partial D^2 \times \{*\}) = \gamma \end{array}$$

Fact: depends on γ only. Reason: f_0, f_1 as above
$\Rightarrow f_1^{-1} \circ f_0 : \partial D^2 \times S^1 \circlearrowleft$ maps $\partial D^2 \times \{*\}$ to itself.
Claim : it extends to self-homeo F of $D^2 \times S^1$.
Implies conclusion :

$$N_0 = M \cup_{f_0} (D^2 \times S^1)$$
$$\Big\downarrow (id, F)$$
$$N_1 = M \cup_{f_1} (D^2 \times S^1)$$

well-def homeo: $x \in T$, $u \in \partial D^2 \times S^1$ same pt in N_0 means $x = f_0(u)$; mapped to $x \in T$, $F(u) \in \partial D^2 \times S^1$; same in N_1 if $x = f_1(F(u))$ i.e. $x = f_1(f_1^{-1} \circ f_0)(u)$ i.e. $x = f_0(u)$.

Existence of F: up to isotopy can assume $f_1^{-1} \circ f_0$ is id or reflection on $\partial D^2 \times \{*\}$ — but reflection extends so wlog $f_1^{-1} \circ f_0 = id$ on $\partial D^2 \times \{*\} \Rightarrow$ can extend it to id on $D^2 \times \{*\}$

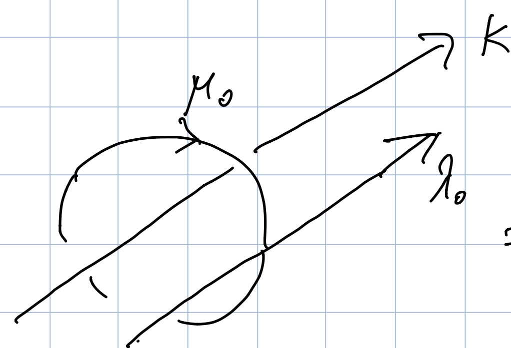


cut $D^2 \times S^1$ along meridian to get D^3; have $S^2 \to S^2$ which extends radially to D^3.

$\gamma \subset T$ slope on T.

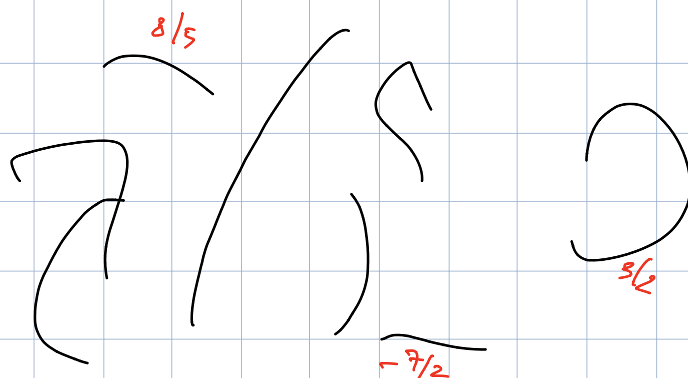
Given $L \subset S^3$ call Dehn surgery along L a Dehn filling of $E(L) = S^3 \setminus \mathring{U}(L)$; $\partial E(L) = T_1 \cup \dots \cup T_k$ all filled to get closed manifold + orientable.

Recall: for knot K privileged basis $H_1(\partial U(K))$



$\pm(\mu_0, \lambda_0)$

$\partial E(L) \supset \partial U(K) \supset \gamma \longleftrightarrow \pm(p\mu_0 + q\lambda_0) \longleftrightarrow p/q$



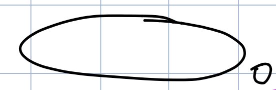
= a 3-manifold

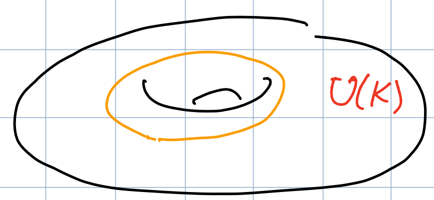
Rem.: integer surgery = surgery along slopes that are longitudes

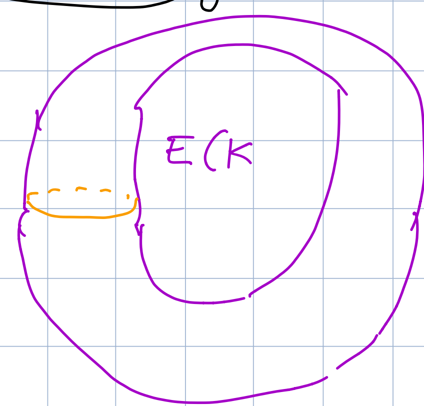
$\rightarrow$ framed link defines a 3-manifold.

Examples: (1) ∞-surgery on K gives S^3

(2) 0-surgery on unknot



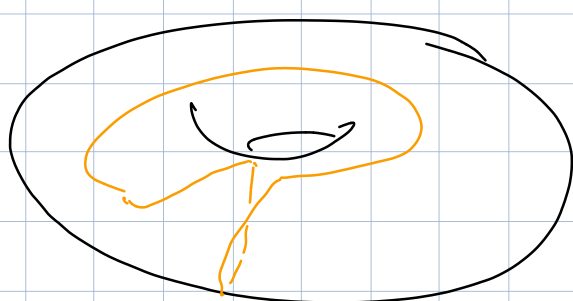


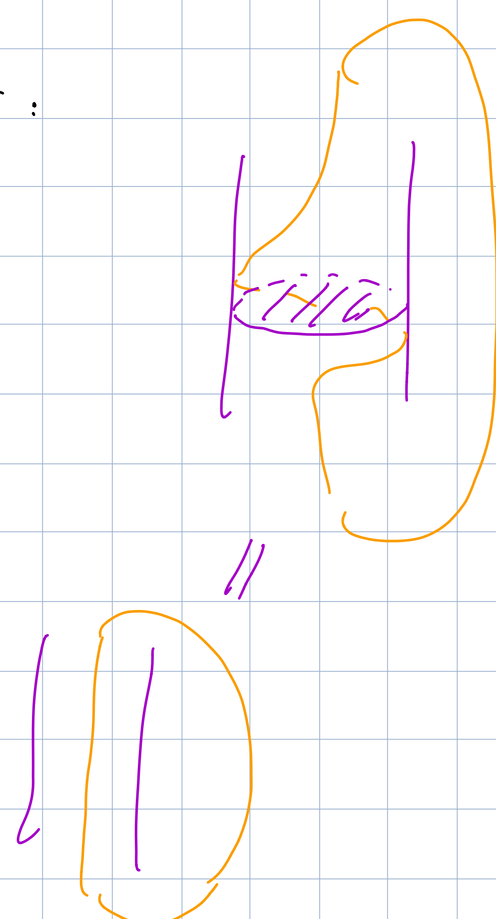


attaching $D^2 \times S^1$ to $D^2 \times S^1$ along id of $\partial D^2 \times S^1$

$\Longrightarrow S^2 \times S^1$.

(1) ± 1 filling of unknot:





$\Longrightarrow S^3$

Attaching a 2-handle to 4-manifold X

$$f : \underbrace{\partial D^2 \times D^2}_{\text{solid torus}} \longrightarrow \partial X$$

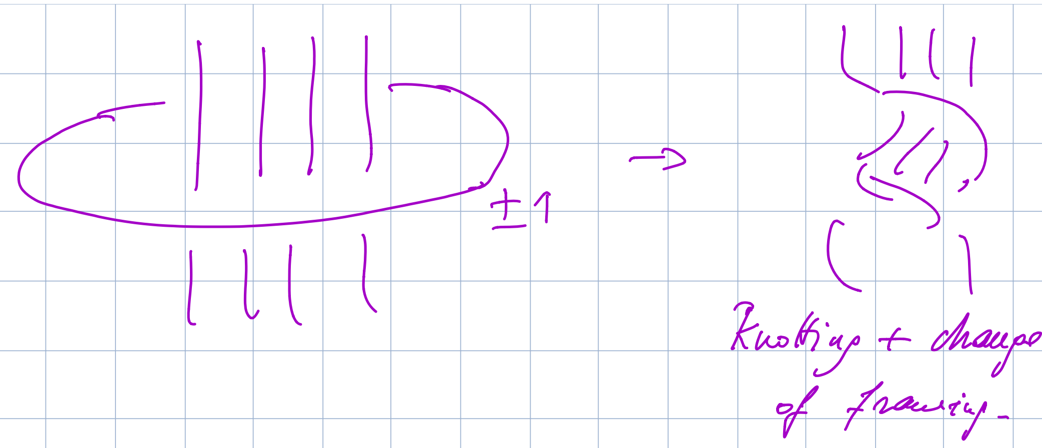
New boundary : $\left(\underbrace{D^2 \times \partial D^2}_{\text{solid torus}}\right) \cup_{f|\underbrace{\partial D^2 \times \partial D^2}_{\text{torus}}} \left(\partial X \setminus \mathrm{Im}(f)\right)$

Exercise: actually integer Dehn surgery (filling curve is longitude).

$\Rightarrow M^3$ presented as surgery along $L \subset S^3$ gives realization of $M = \partial\left(D^4 \text{ with 2-handles attached}\right)$

$\Rightarrow M = \partial\left(\text{singly connected 4-mfld}\right)$

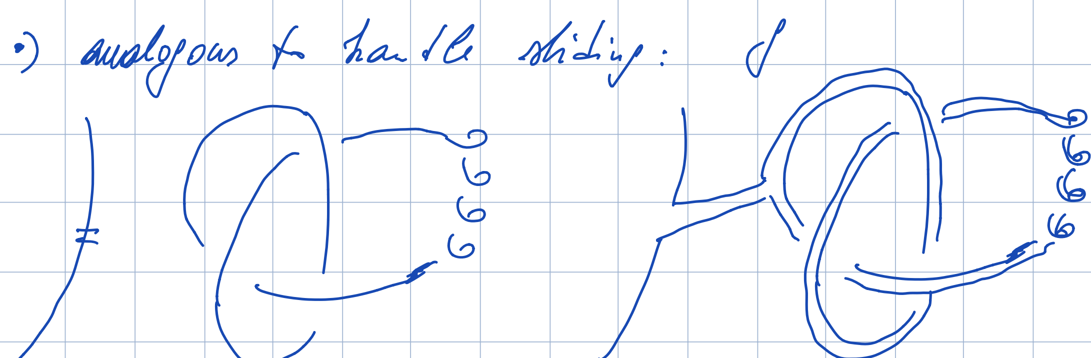
Thm (Rokhlin-Lickorish):
every closed orientable $M^{(3)}$ can be presented by integer surgery along $L \subset S^3$
(actually: with individual components unknotted & coefficients ± 1)



Fact: Kauffman bracket is invariant for framed links; using it + extensions can get "quantum" invariants of 3-manifolds using an approach similar to knot invariants through to Kirby moves that relate different presentations of the same manifold.

Kirby moves: •) stabilization adding  ± 1 as above
(the associated 4-manifold is changed as $X \rightsquigarrow X \# (\pm \mathbb{P}^2(\mathbb{C}))$)

•) analogous to handle sliding:



(Martelli: local version of Kirby moves)

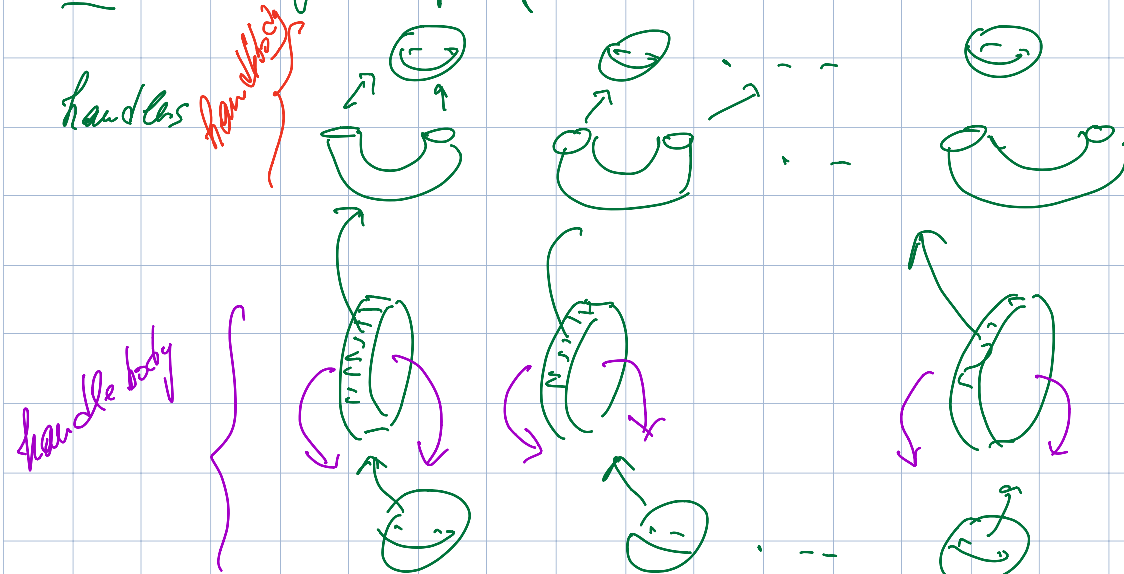
—— o ——

$\exists$ many proofs of existence of surgery realization.

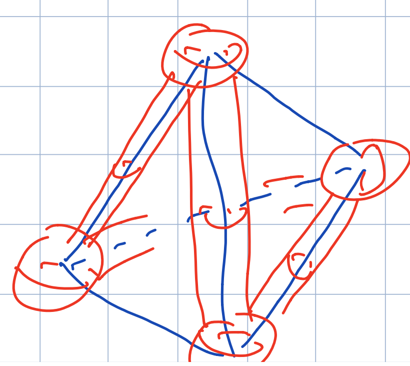
Heegaard splitting of closed orientable $M^{(3)}$ is

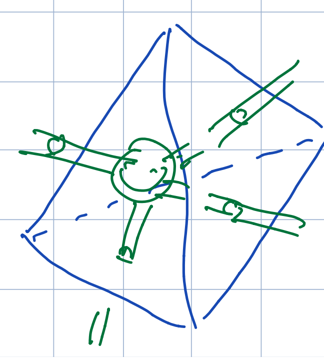
- $\Sigma \subset M^{(3)}$ surface of genus g s.t. cutting M along Σ get two handlebodies
- $M = H_0 \cup_f H_1$ $\quad H_0, H_1$ handlebodies, $f: \partial H_1 \to \partial H_0$.

Fact: Heegaard splittings exist:



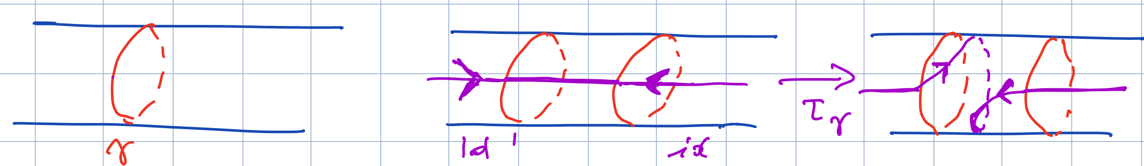
Also PL proof:





Existence of Heegaard splittings shows 3-manifold theory is contained in the study $Aut^+(\Sigma_g)$ (self-homeos / isotopy).

Dehn twist: Σ surface oriented, cpt but possibly $\partial\Sigma \neq \emptyset$.
$\gamma \subset \Sigma$ s.c.c. $\gamma \cap \partial\Sigma = \emptyset$.



well-def / isotopy for unoriented γ

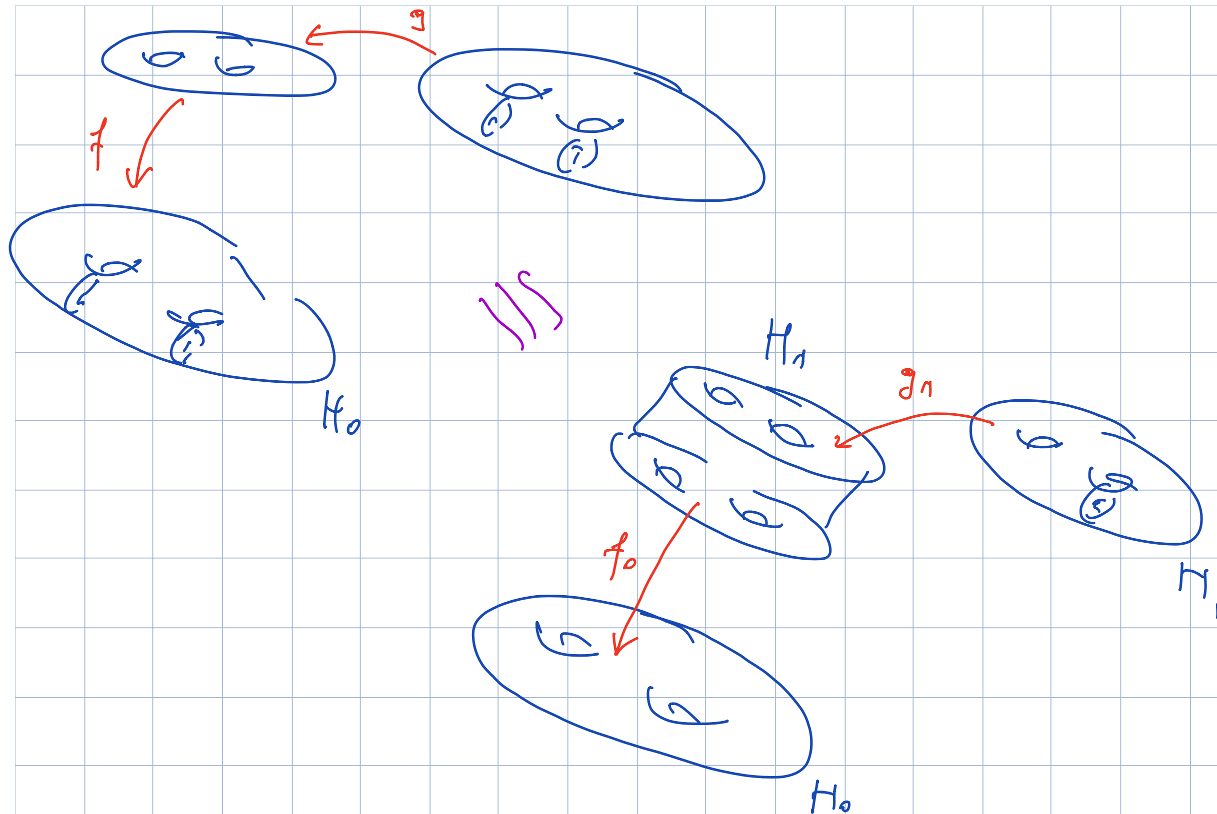
Thm: $Aut^+(\Sigma)$ is generated by Dehn twists.

Assuming this I show existence.

Prop: H_0, H_1 handlebodies, Σ surface all same genus
$g : \partial H_1 \to \Sigma$, $f : \Sigma \to \partial H_0$

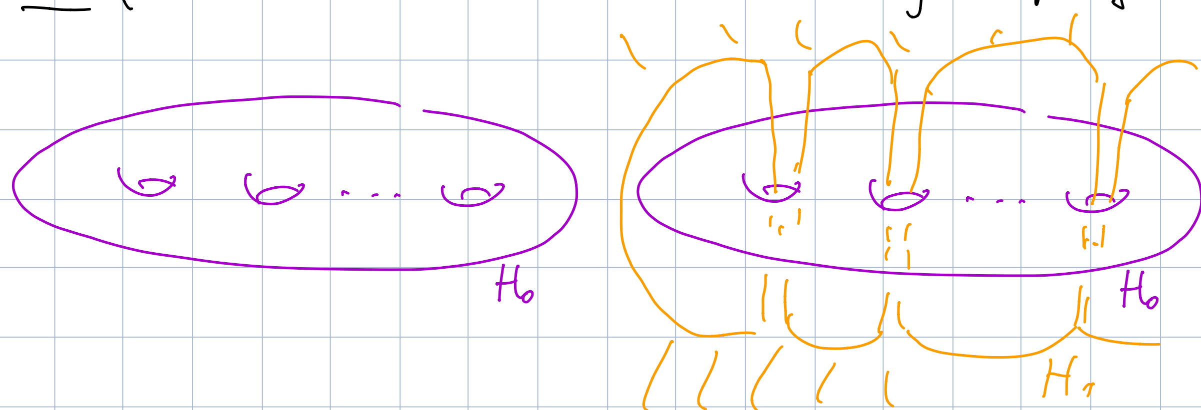
$$\Rightarrow H_0 \cup_{f \circ g} H_1 \cong H_0 \cup_{f_0} (\Sigma \times [0,1]) \cup_{g_1} H_1$$

$f_0 : \Sigma \times \{0\} \to \partial H_0 \quad (x,0) \mapsto f(x)$
$g_1 : \partial H_1 \to \Sigma \times \{1\} \quad x \mapsto (g(x), 1)$



(similar to first proof)

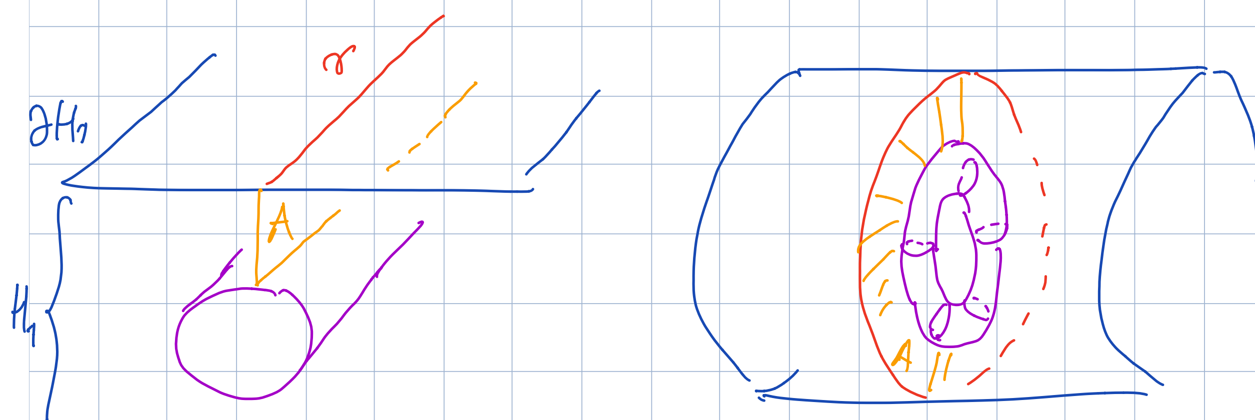
Proof: take $S^3 = H_0 \cup H_1$ trivial Heegaard splitting

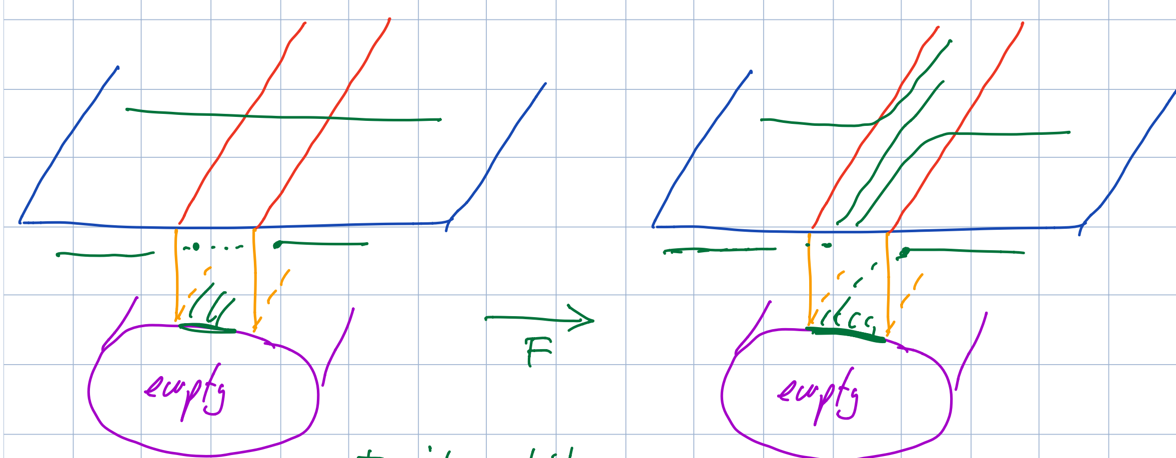


Want to produce surgery presentation starting from $M = H_0 \cup_f H_1$

Suppose first $M = H_0 \cup_{\tau_\gamma^{\pm 1}} H_1$. $\quad \tau_\gamma : \underset{\partial H_0}{\overset{\|}{\partial H_1}} \to \underset{\partial H_1}{\overset{\|}{\partial H_0}}$

Take parallel copy γ' of γ pushed inside H_1 and U neighborhood of γ'; claim: $\tau_\gamma : \partial H_1 \to \partial H_1$ extends to $F : H_1 \setminus U \to H_1 \setminus U$.





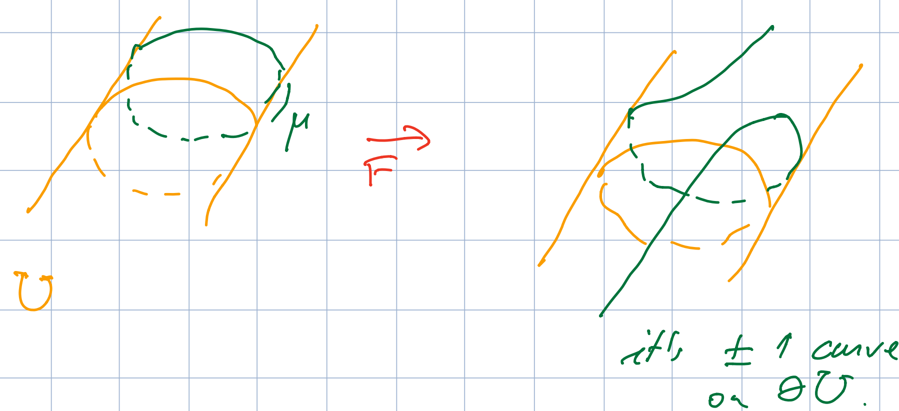
$F = id$ outside $A \times [-1,1]$

On $A \times [-1,1]$ acts as on $\gamma \times [-1,1]$

$$
\begin{array}{ccccc}
M \setminus U & = & H_0 & \cup_{\tau_\gamma} & (H_1 \setminus U) \\
\downarrow & & u \mapsto u & & x \mapsto F(x) \\
S^3 \setminus U & = & H_0 & \cup_{id} & (H_1 \setminus U)
\end{array}
$$

well-defined homeo $\Rightarrow$ M obtained by Dehn filling $S^3 \setminus U$ along curve that is the image under F of meridian of U in H_1 :

<u>Statement</u> : can take all surgery coeffs ± 1 on all components of L trivial.



Now take $M = H_0 \cup_f H_1$ $\quad f = \tau_0 \circ \dots \circ \tau_n$

$$\tau_j = \tau_{\gamma_j}^{\varepsilon_j} \qquad \varepsilon_j = \pm 1$$

Using repeatedly technical proposition see :

$$M = H_0 \cup_{\tau_0^{(0)}} \left(\Sigma \times [0,1]\right) \cup \dots \cup \left(\Sigma \times [n-1,n]\right) \cup_{\tau_n^{(n)}} H_1$$

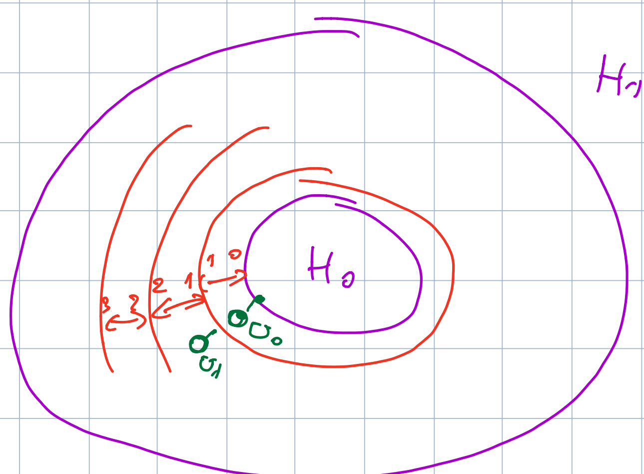
Repeating what done for one twist :

$$M \setminus (U_0 \cup \dots \cup U_m) =$$

$$= H_0 \cup_{\tau_0^{(0)}} \left(\Sigma \times [0,1] \setminus U_0\right) \cup_{\tau_1^{(1)}} \dots \cup \left(\Sigma \times [0,1] \setminus U_{m-1}\right) \cup_{\tau_m^{(m)}} \left(H_1 \setminus U_m\right)$$

$$\begin{array}{ccc} H_0 & & \left(\Sigma \times [0,1] \setminus U_0\right) \\ {\scriptstyle id}\downarrow & & \downarrow {\scriptstyle F_0} \\ H_0 & \cup_{id} & \left(\Sigma \times [0,1] \setminus U_0\right) \cup_{id} \dots \cup \end{array}$$

F_0 extension to $\tau_0^{(0)}$ as above
map well-def homeo and target
is $S^3 \setminus (U_0 \cup \dots \cup U_m)$



each U_j is knot neighbourhood in S^3 & curve to Dehn fill on it to get M is a $\pm$ longitude privileged (as above).

Wlog : can suppose components individually unknotted:

Thm: $Aut^+(\Sigma, \partial\Sigma)$ is $\mathcal{D}$ = its subgroup generated by Dehn twists.

Proof: by induction on $g(\hat{\Sigma})$
($\hat{\Sigma}$ = closed surface obtained from Σ ...)

Basis of induction: $g=0$, Σ = punctured sphere
long proof using braid theory omitted.

Inductive step: $g>0 \Rightarrow \exists\, \alpha \subset \Sigma$ non-sep. s.c.c.
Take $f \in Aut^+(\Sigma)$, consider $\alpha, f(\alpha)$.

Key argument: show that $\exists\, d \in \mathcal{D}$ s.t. $d(\alpha) = f(\alpha)$.

Conclusion: $(d^{-1} \circ f)(\alpha) = \alpha$; with work can assume

$\alpha^{-1} \circ f$ is id on α

$\Rightarrow$ can cut Σ along α getting smaller genus with $\alpha^{-1} \circ f$ well-def on it ; apply induction whence conclusion.