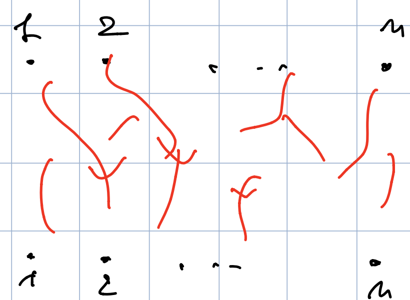
Teoria dei Nodi

9/5/2019

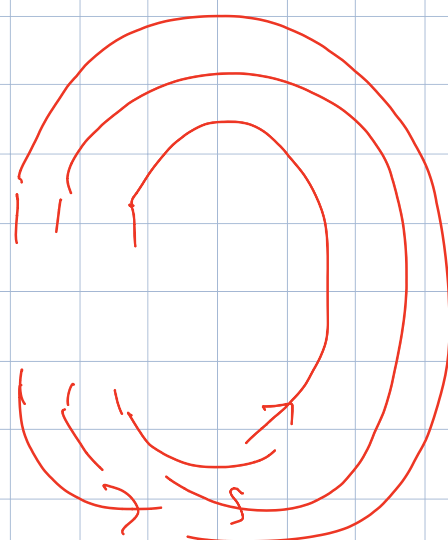
Riempire questionari sul corso.

$B_n =$



/ isotopy through same.

$B_n \ni \beta \longmapsto \hat{\beta}$

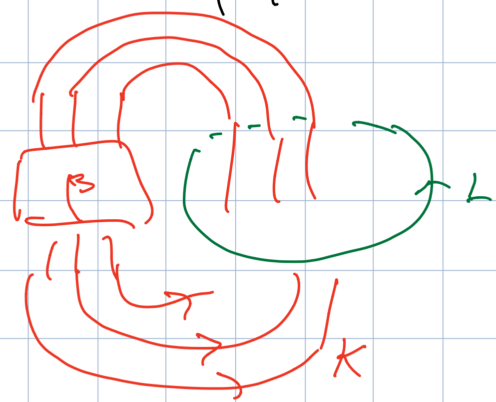


<u>Thm</u>: $\forall$ oriented link L $\exists$ n, $\beta \in B_n$ s.t. $\hat{\beta} = L$.

<u>Thm</u>: $\hat{\beta}_1 \cong \hat{\beta}_2$ iff β_1, β_2 Markov equiv.

- conjugation
- stabilization $B_n \ni \beta \longmapsto \beta \cdot \sigma_n^{\pm 1} \in B_{n+1}$.

Def: complete closure of β



Def: braided link: oriented $K \cup L$ s.t. $L \cong$ unknot

$S^3 \setminus L \cong B^2 \times S^1$, $p_L : S^3 \setminus L \to S^1$

p_L monotonic increasing on K.

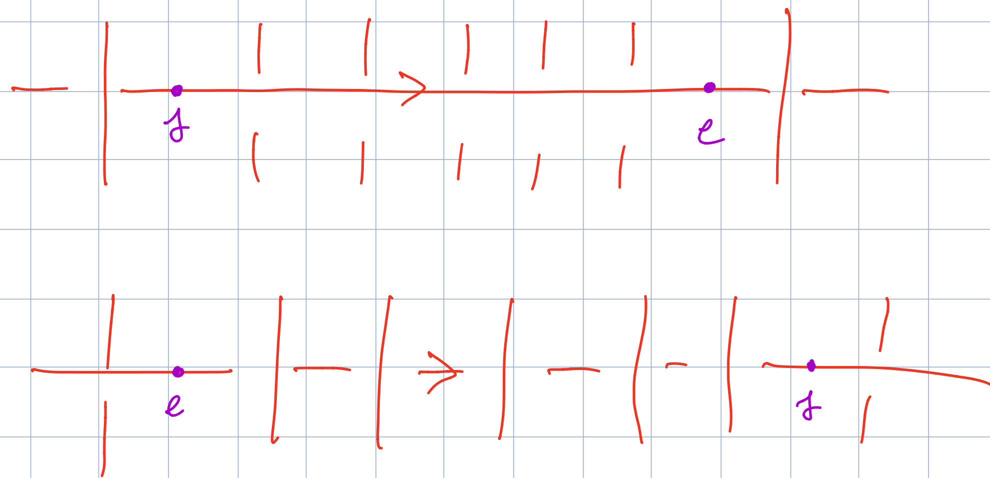
Rem: complete closure is braided link.

Prop: braided links $/$ isotopy $=$ braids $/$ conjugation.

Prop: if for $K \cup L$ we have $p_L|_K$ non-decreasing + non-constant on any compo $\Rightarrow$ braided / isotopy.

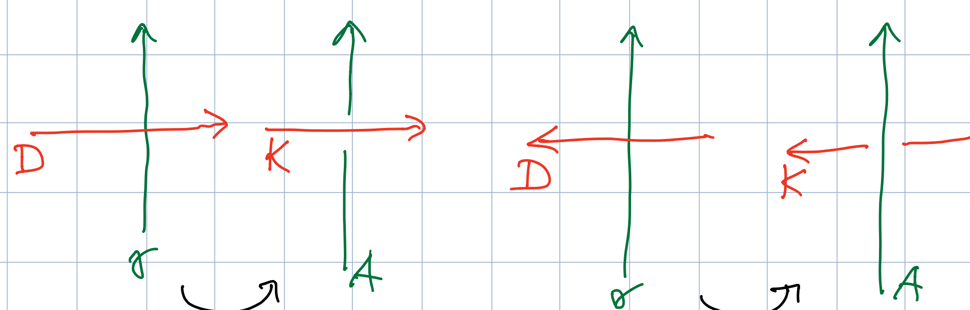
Threading:

- choice of overarcs for diagram D is $S, E \subset D$ s.t. all $[s,e] \subset D$ contain overcrossings only, all $[e,s] \subset D$ undercrossings only (maybe more).



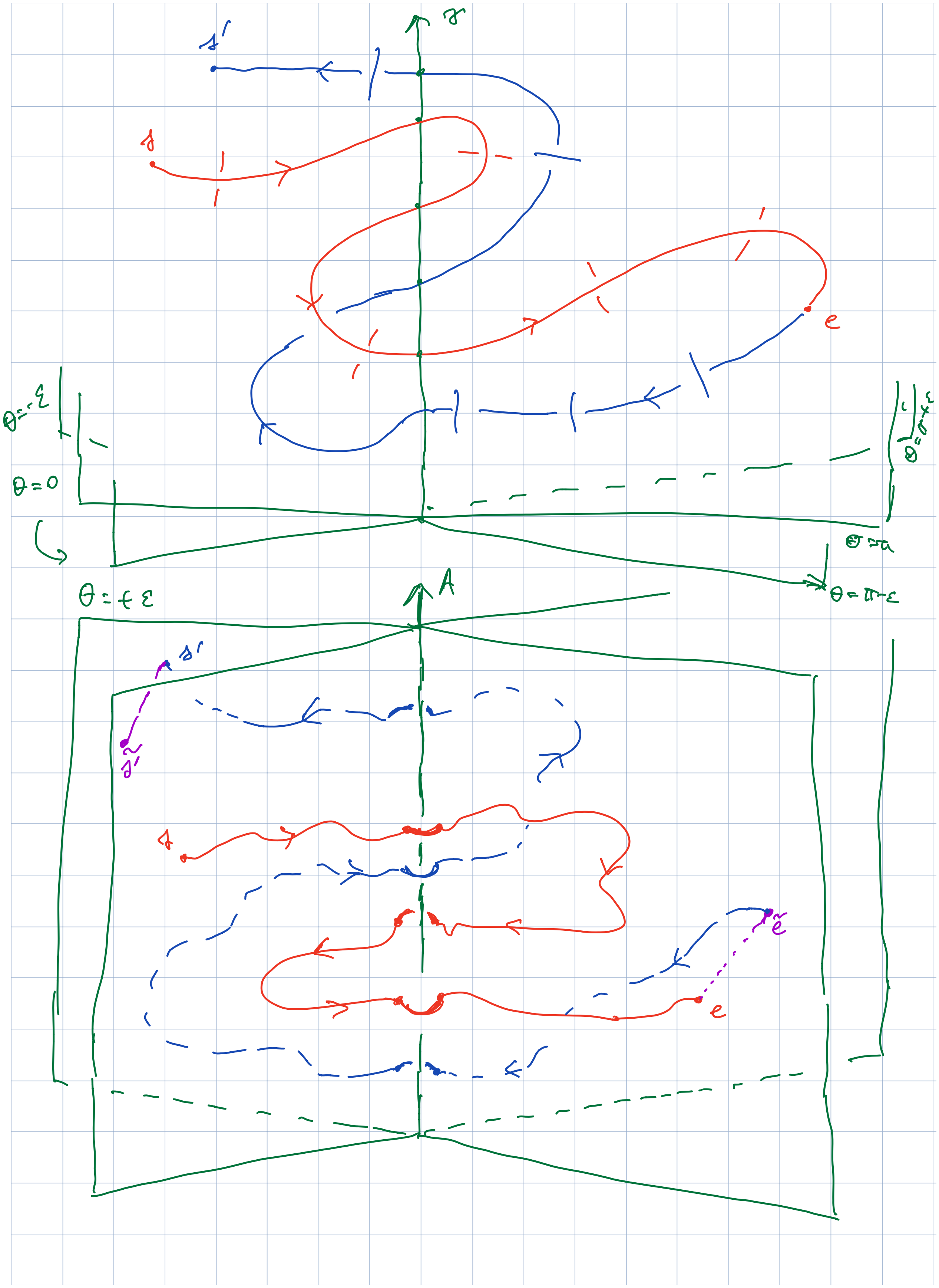
Given D, S, E call threading this process:

- choose $\gamma \subset \mathbb{R}^2$ loop oriented that separates S from E leaving S to its left
  γ transversal to D
- turn $D \cup \gamma$ into link $K \cup A$



Prop: such a K, A is braided link ($\Rightarrow$ Alexander).

Proof: isotope γ to be straight line in $\mathbb{R}^2$



γ
γ'
γ
e
θ = -ε
θ = 0
θ = π+ε
θ = π
θ = π-ε
θ = +ε
A
γ'
γ̃'
γ
ẽ
e

overarcs are all contained in $\theta = +\varepsilon$ & $\theta = \pi - \varepsilon$

except 

underarcs are all contained in $\theta = -\varepsilon$ & $\theta = \pi + \varepsilon$

except 

Since $\theta/_K$ is non-deg + not constant on any compo, it's braided. $\blacksquare$

Remark: Markov moves extend to the complete closure of a braid $\Rightarrow$ makes sense to speak about Markov equivalence for braided links.

Markov's thm follows from these:

Thm 1: the complete closure of $\beta \in B_n$ is the threading of a diagram of $\hat{\beta}$.

Thm 2: (a) any two threadings of same diagram are Markov equivalent

(b) two diagrams of isotopic oriented links are Markov equivalent.

$\widehat{\beta}_1 \cong \widehat{\beta}_2$ choose D_1, D_2 s.t.
$\widehat{\beta}_j$ is threading of D_j (Thm 1)

all A = complete closure

$\exists$ threadings $(D_1, A_1), (D_2, A_2)$ that are Markov-equiv (Thm 2(b))

but by Prop. $(D_j, A_j) = \widehat{\beta_j'}$ hence

$\widehat{\beta_1'}, \widehat{\beta_2'}$ are Markov equiv;

and $\widehat{\beta_j}, \widehat{\beta_j'}$ are Markov equi (Thm 2(a))

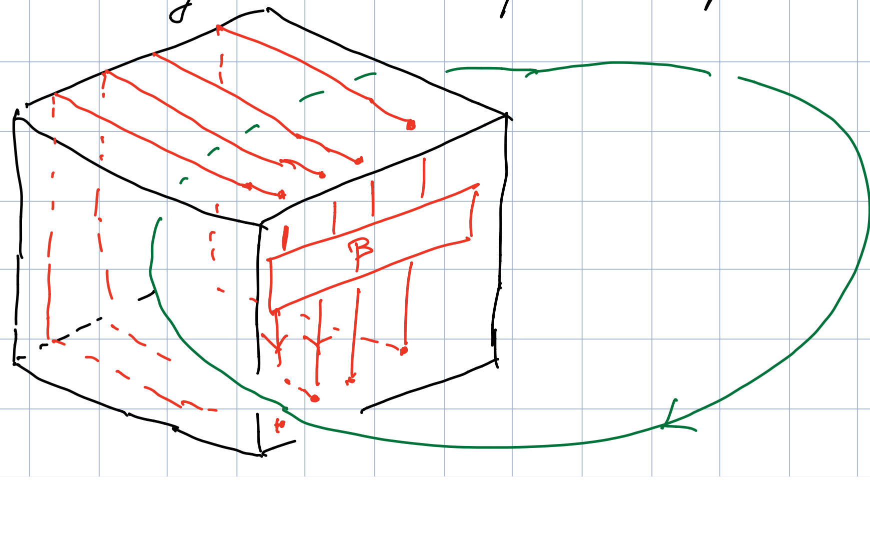
$\Longrightarrow \widehat{\beta_1}, \widehat{\beta_2}$ Markov equiv

$\Rightarrow \beta_1, \beta_2$ Markov equiv.

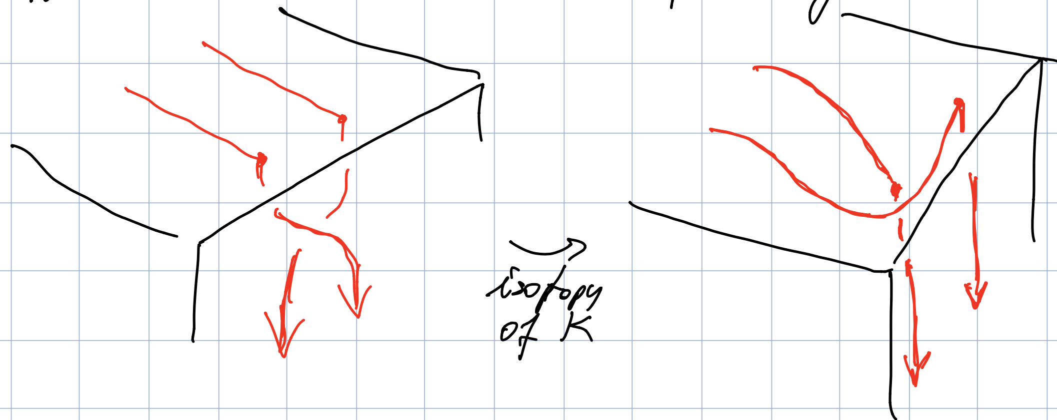
<u>Thm 1</u>: given any β, $\widehat{\beta}$ is threading of some diagram of $\widehat{\beta}$.

complete closure

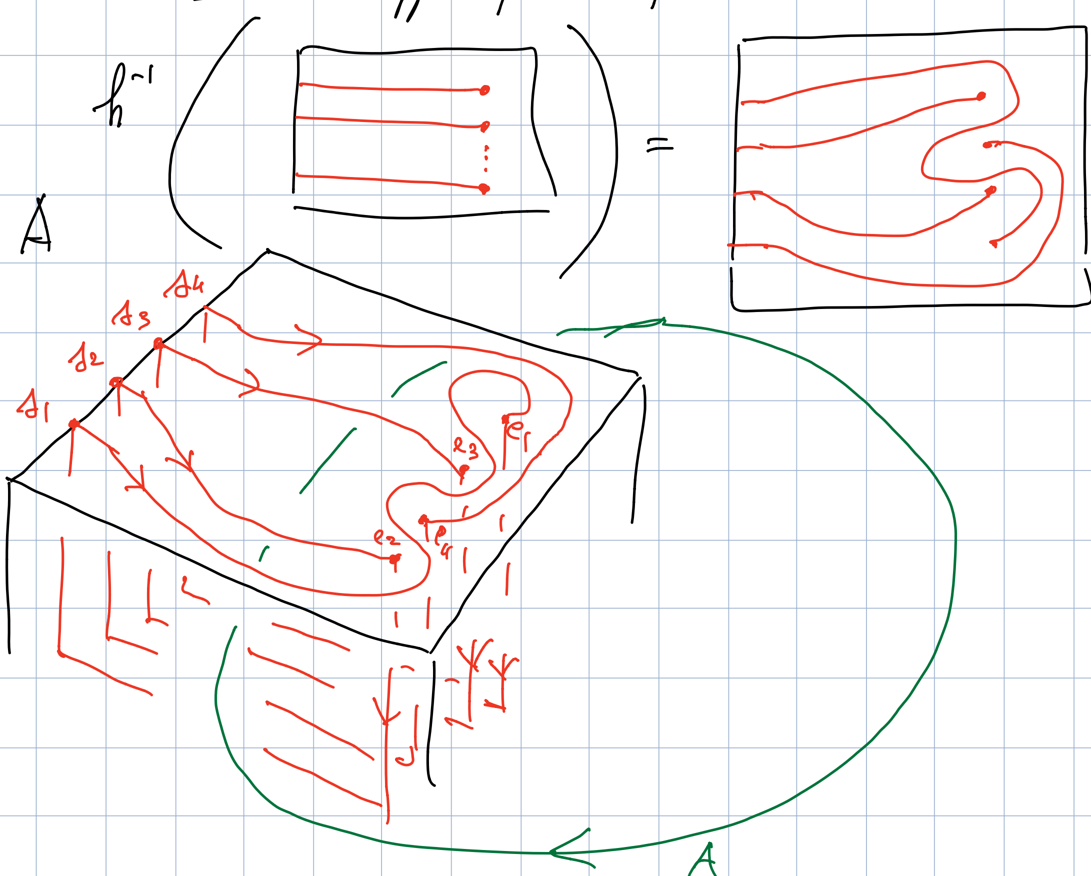
<u>Proof</u>: view closure of β as being drawn in a cube almost entirely on the surface except where β is:

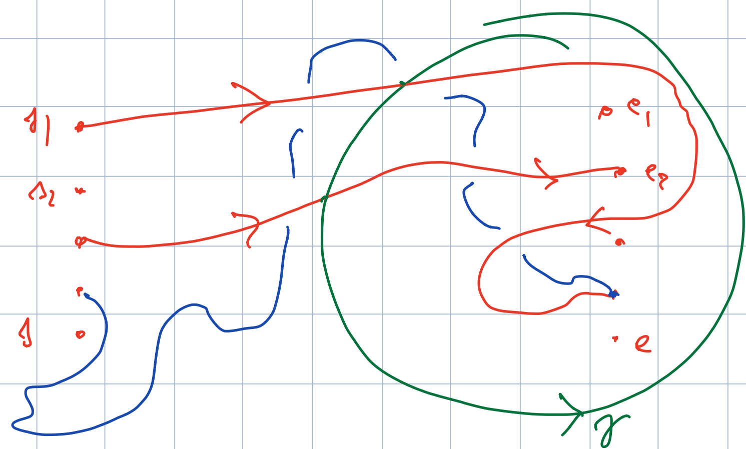


isotope on the surface of the upper face of the cube the strands so to make β straight :



if β is viewed $h:(D^2,\{1,\dots,n\})\circlearrowleft$ then the strands on upper face of cube are



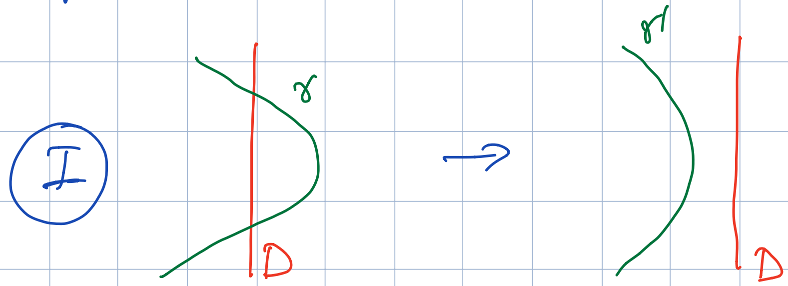


Exercise: convince yourself that threading of this is the complete closure of β. ☒

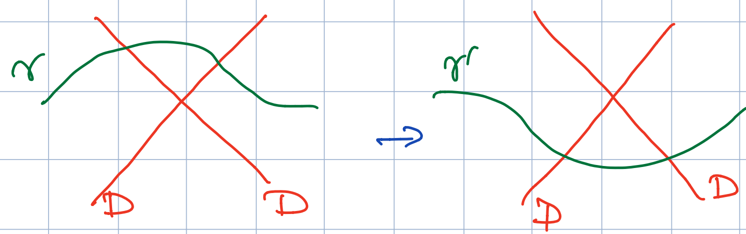
Thm 2(a): two threadings of same diag. are Markov-equiv.

Proof:

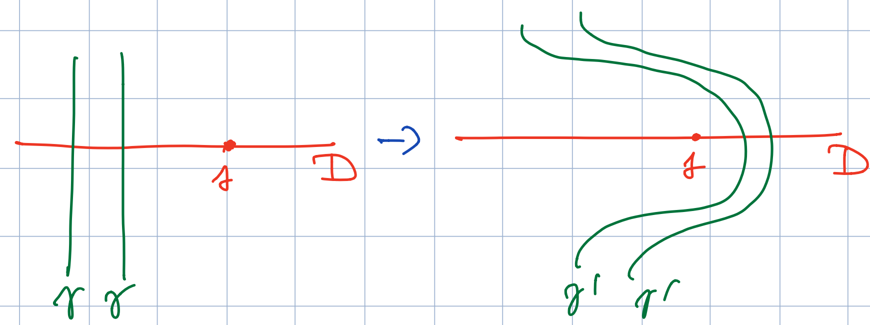
Claim 0: given S, E choice of overarcs for D if γ, γ' are curves separating S from E leaving S to the left then they are related by these moves:



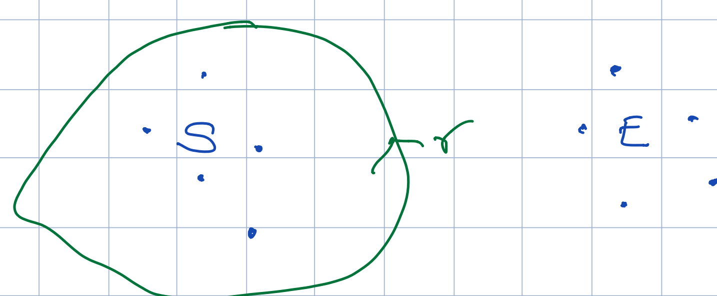
II



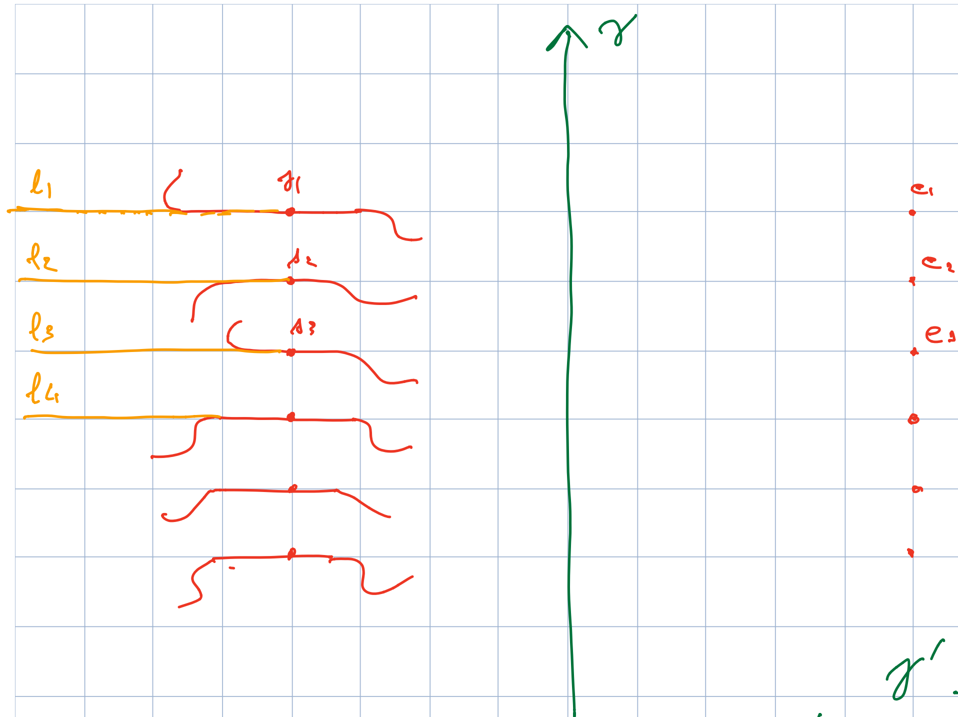
III



I + II generate isotopy rel. $S \cup E$



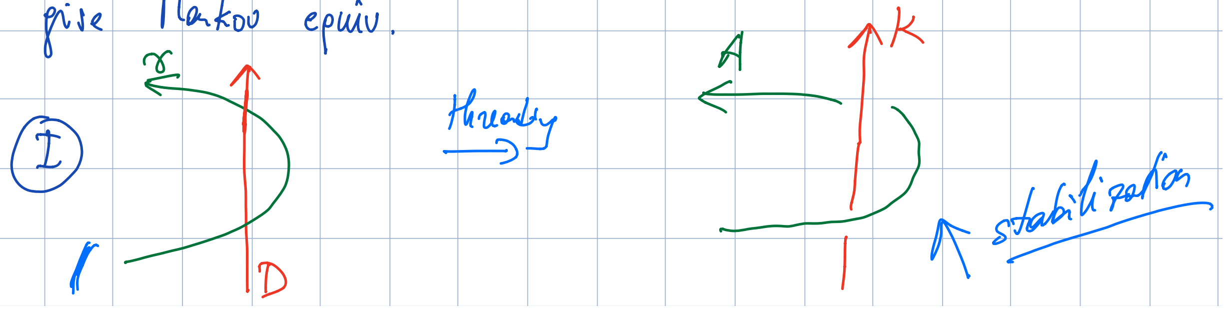
Must show that using III we can transform γ' to a curve isotopic to γ rel. $S \cup E$. In fact: can isotope so that:

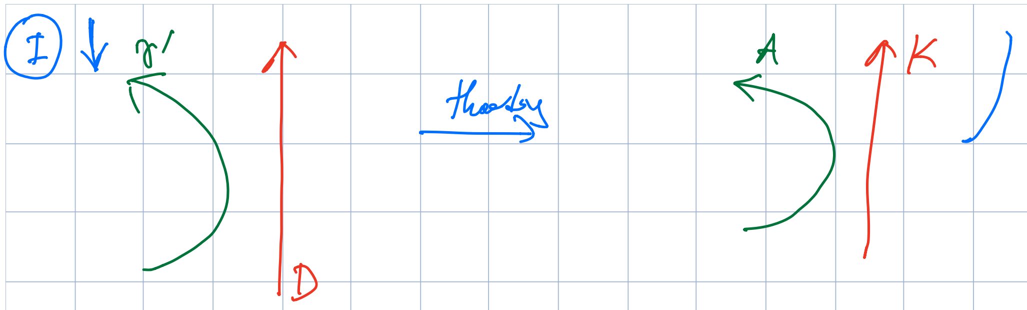


γ' bounds a bounded disc containing E whence it crosses each l_j an even number of times

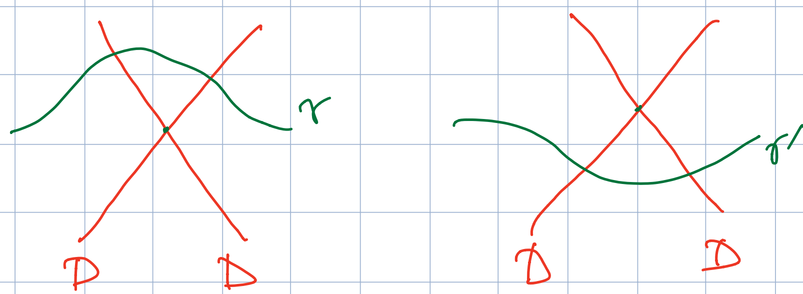
$\Rightarrow$ using III can assume γ' disjoint from all l_j's
$\Rightarrow$ isotopic to γ rel. $S \cup E$. $\square$

Claim 1: if (K,A), (K,A') are threadings constructed from different γ, γ' for the same S, E then are Markov-equiv. Must show I, II, III above give Markov equiv.

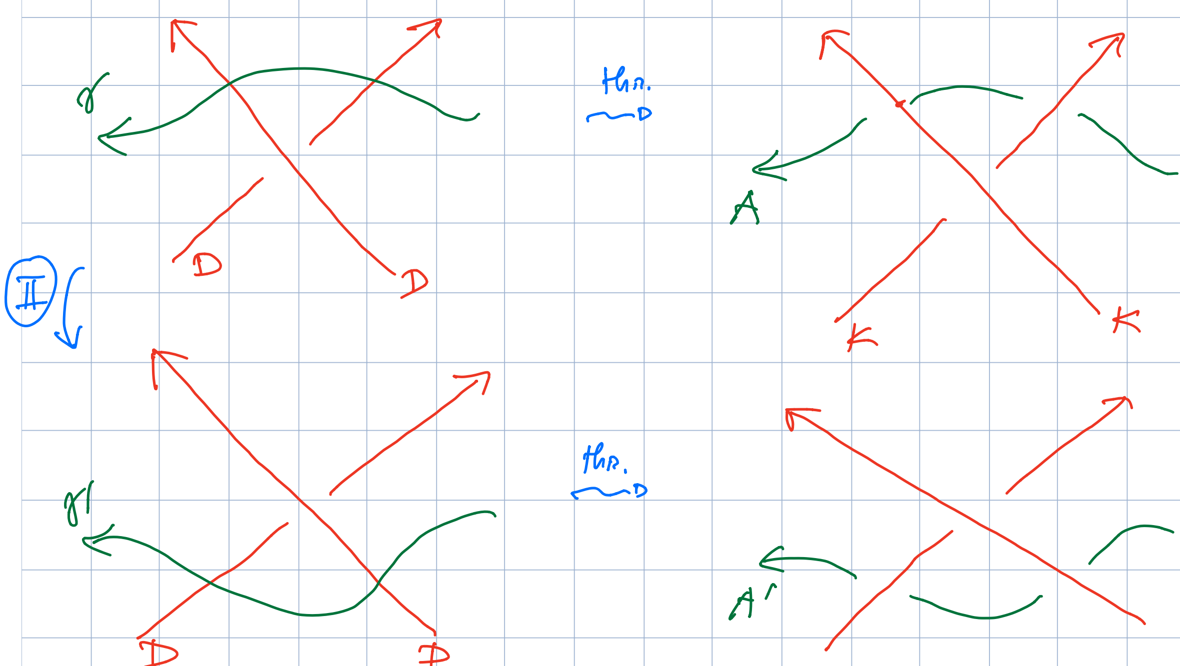


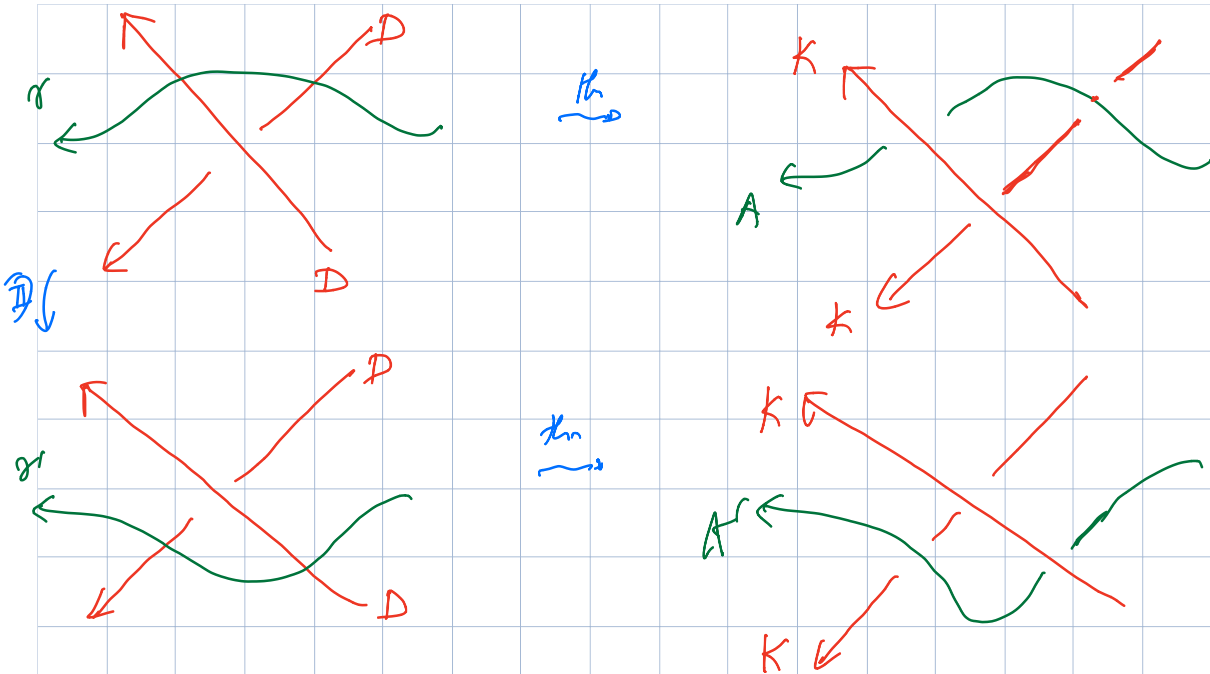


Type II

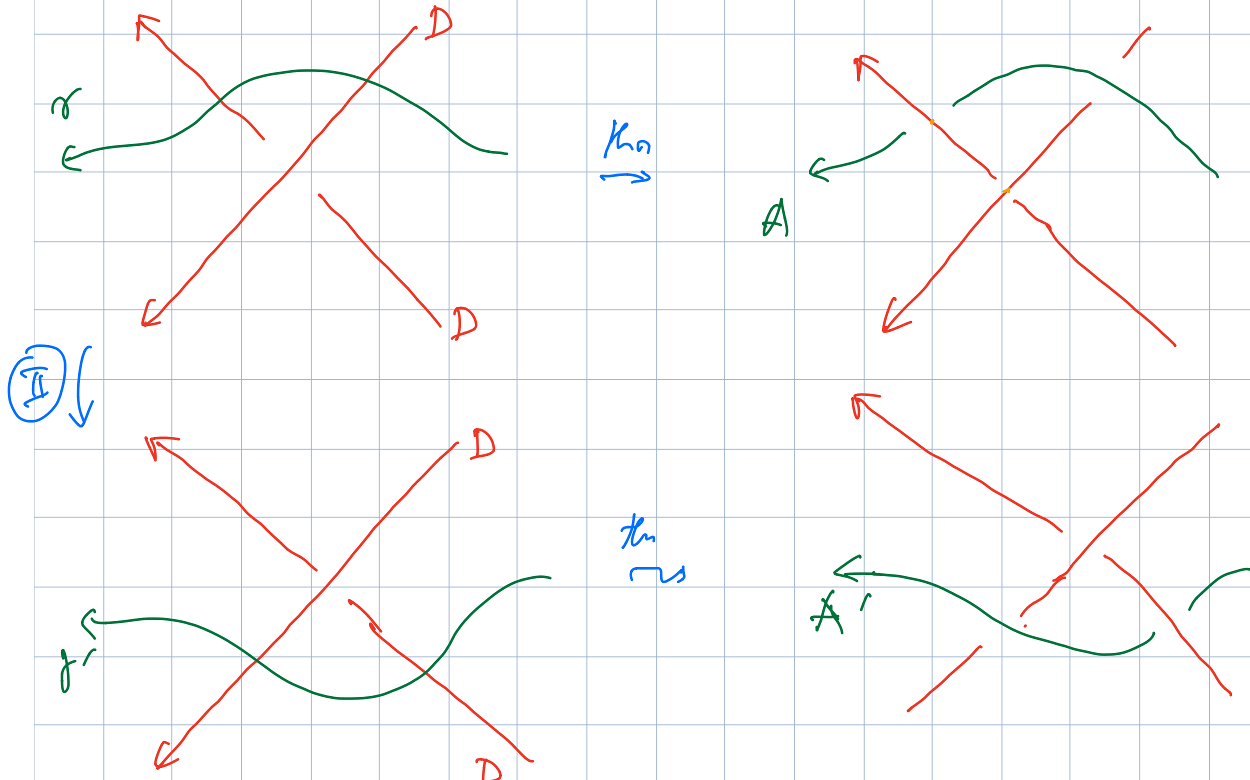


16 possible configs for type of crossing + orientations.
Some give isotopic theadings:



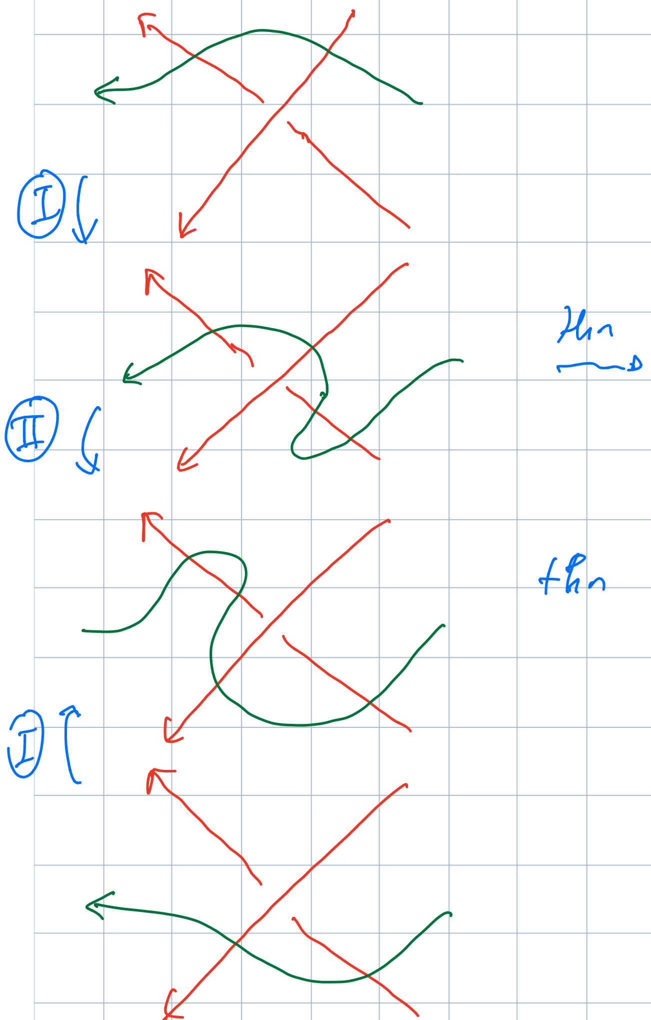


In some cases it's <u>not</u> isotopy :

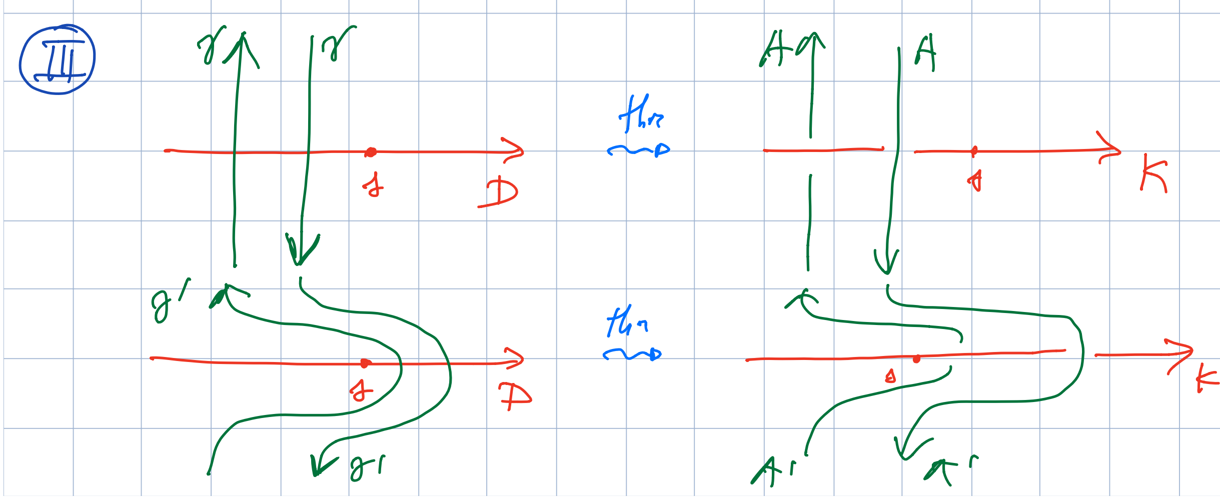


Fact : when II does not give isotopy I can

factor it through type I + II's giving isotopy.
For the previous example:



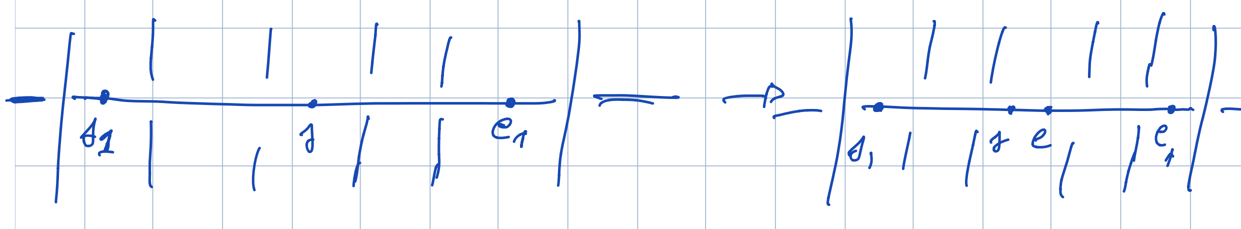
Exercise: draw picture & convince yourself that it's isotopy.



this is isotopy.

Claim 2: if (S,E) choice of overarcs for D
$s \in D$, $s \notin S \cup E \cup \{\text{crossings}\}$
$\Rightarrow \exists\, (\tilde{S},\tilde{E})$ s.t. $\tilde{S} \supset S \cup \{s\}$.

Proof: s can belong to overarc or underarc



same if on underarc.

Conclusion: given (S,E), (S',E') up to small perturbation can assume all S, E, S', E' mutually disjoint. By Claim 2 (+ analogue for e) $\exists\ \tilde{S},\tilde{E}$ choice of overarcs s.t.
$\tilde{S} \supset S \cup S'$ $\quad E' \supset E \cup E'$.
If choose γ separating $\tilde{S}$ from $\tilde{E}$ also separates S from E, S' from E'

but threading depends on γ only so by claim 1 any threading for (S,E) and for

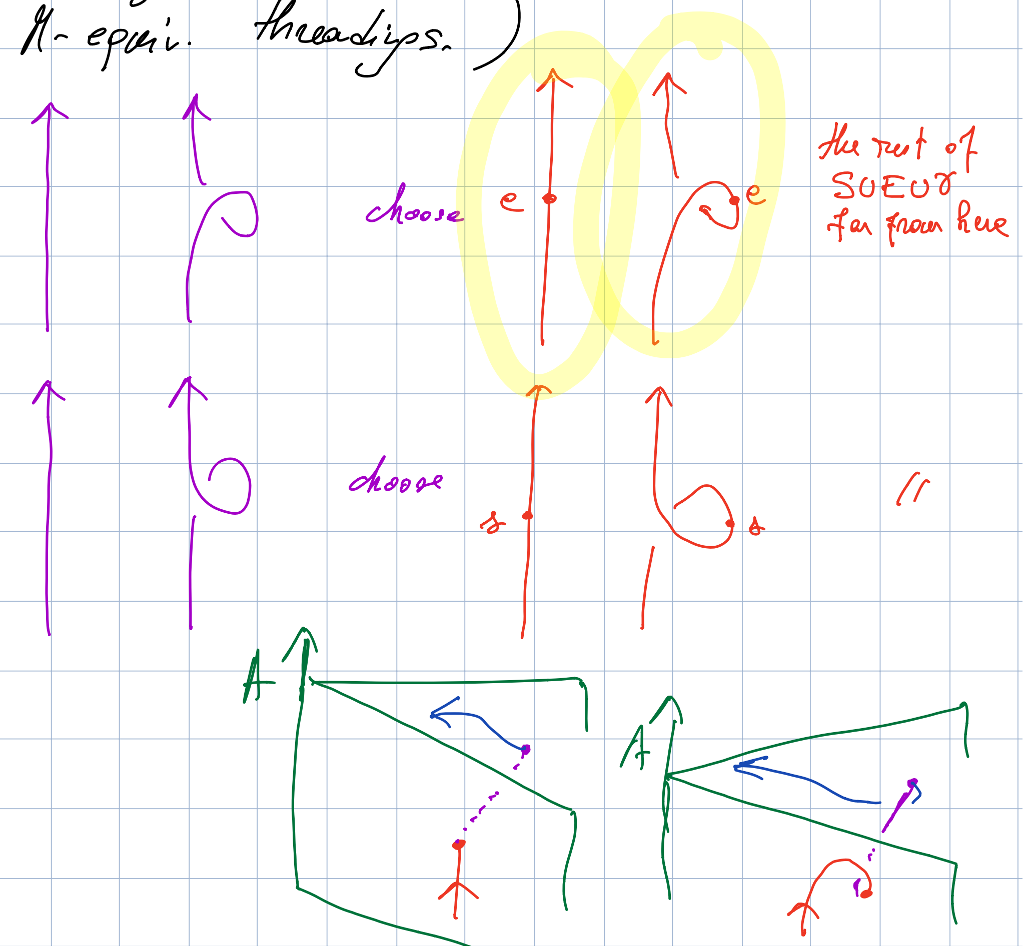
(S',E') is M-equiv. to threading given by γ. ∎

<u>Thm 2(b)</u>: two diagrams of isotopic links have M-equiv. threadings.

<u>Proof</u>: we actually show that if D, D' are related by one Reidemeister move then for suitable choice of S, E, S', E' and of γ, γ' the threadings are actually isotopic.

(This suffices by Thm 2(a) because different choices give M-equiv. threadings.)

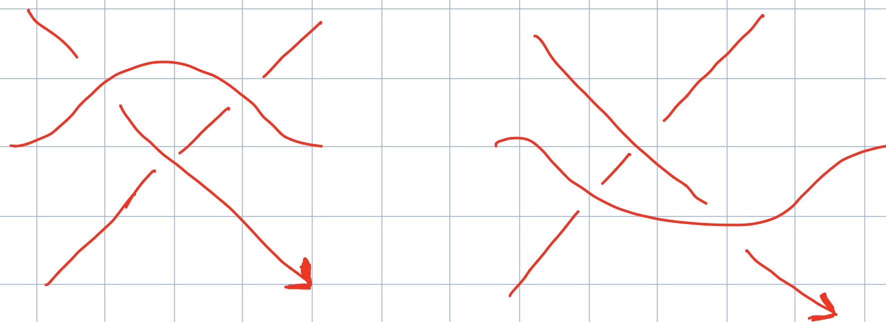
R_I:



isotopic

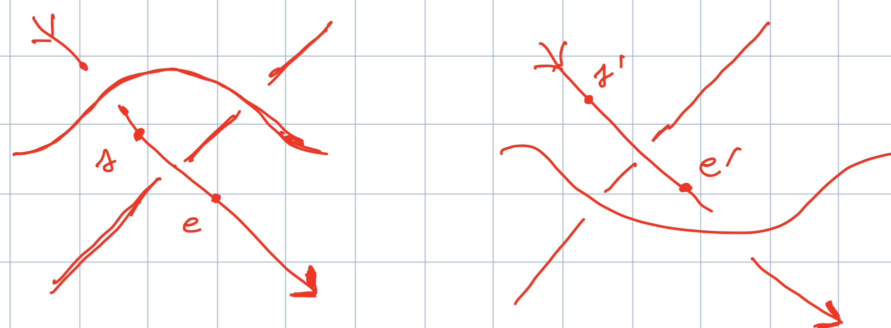
R_{II} : choose all of $S \cup E \cup \gamma$ far from move.

R_{III} :

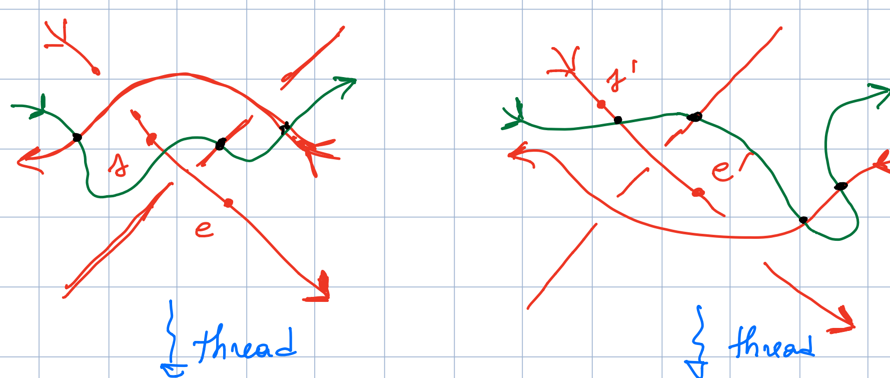


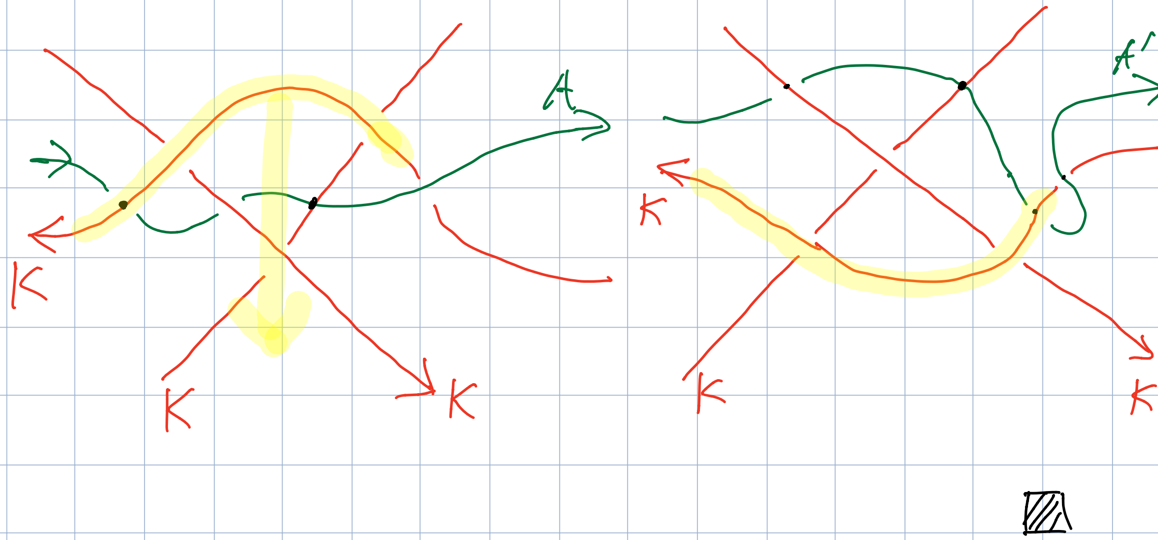
orientation of middle arc can be chosen like this up to reversing move.

choice of S, E, S', E'



choice of γ, γ' depends also on orientation of arcs:





A
K
K
K
A'
K
K
K