Knot theory 7/5/2019

We are constructing (using induction on the number of crossings of diagrams) a polynomial invariant

$$P : \left\{ \begin{array}{c}\text{oriented links} \\ \text{in } S^3\end{array} \right\} \to \mathbb{Z}[\ell^{\pm 1}, m^{\pm 1}].$$

We defined ascending diagrams (w.r.t some choices). In defining P on the set of diagrams with $\leq n$ crossings, we agreed that, for any diagram D, we first construct an ascending diagram D_α, and we compute $P(D)$ starting from D_α, for which

$$P(D_\alpha) = \mu^{\#D - 1}$$

and repeatedly using the skein formula at every crossing we need to change.

We already proved:

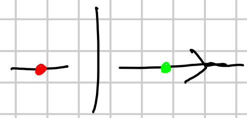
0) this construction does not depend on the order of the crossing changes from D_α to D.

1) the skein formula holds.

We need to prove:

2) The definition does not depend on the choice of the basepoints

Proof: It is sufficient to show I can move a basepoint across a crossing



If the components appearing at the crossing are distinct, then the ascending diagrams associated with these distinct basepoints are the same.

Otherwise, the ascending diagrams associated to these basepoints only differ at this crossing. In this case, the two ascending diagrams D_α and D_β are such that

$$\ell P(D_\alpha) + \ell^{-1} P(D_\beta) + m\, P(\,)(\,) = 0$$

Suppose we defined P using D_α as the preferred ascending diagram. Then:

$P(D_\alpha) = \mu^{\#D-1}$

By desingularizing D_α, we obtain an ascending diagram with $(n-1)$ crossings, and $\#D+1$ components (this uses that at the crossing only one component is appearing). By induction,

$P(\,)(\,) = \mu^{\#D}$, and this implies (using the skein relation)

$$P(D_\beta) = \mu^{\#D-1}$$

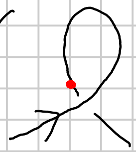
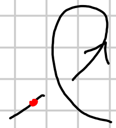
$$\ell P(D_\alpha) + \ell^{-1} P(D_\beta) + m P(\,)(\,) = 0$$

$$\ell \mu^{\#D-1} + \ell^{-1} x + m \mu^{\#D} = 0$$

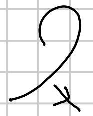
has $x = \mu^{\#D-1}$ as a solution $\left(\mu = -\frac{\ell+\ell^{-1}}{m}\right)$.

This implies (2), because by replacing D_α with D_β (or viceversa) as the starting point to define P on D, I can just go back to D_α (or D_β) and run the previous algorithm, without changing the outcome.

3) P is invariant w.r.t. Reidemeister moves (between diagrams with $\leq n$ crossings).

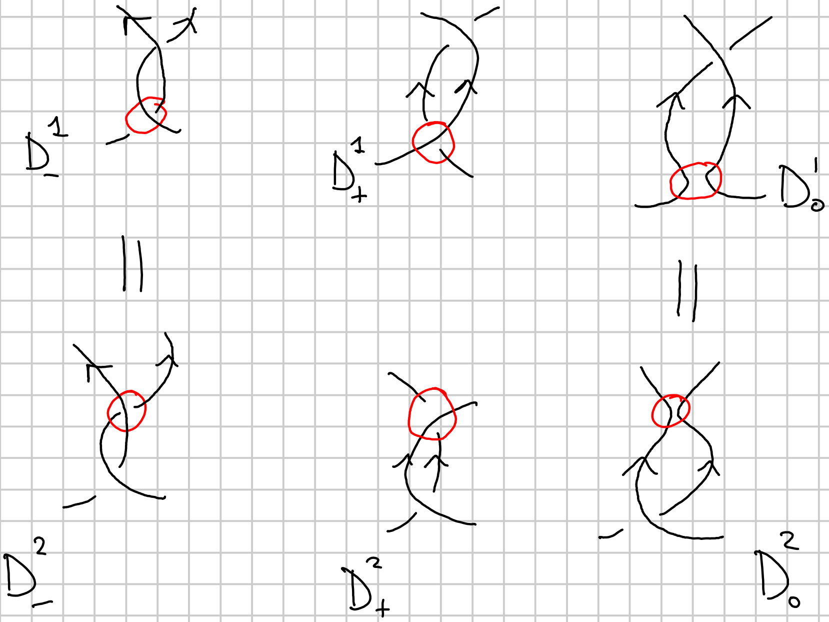
I: P() = P(—) = P()

By (2) I may put the basepoint right before the crossing. With this choice, the ascending diagram coincides with D at the crossing I am looking at. This means that the algebraic computations needed to compute

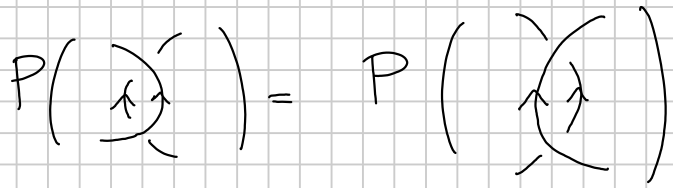
P() and P($\longrightarrow$)

are exactly the same.

II

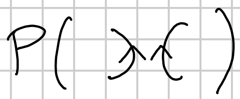


By the skein formula,

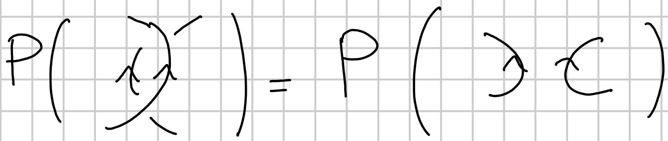


(*)

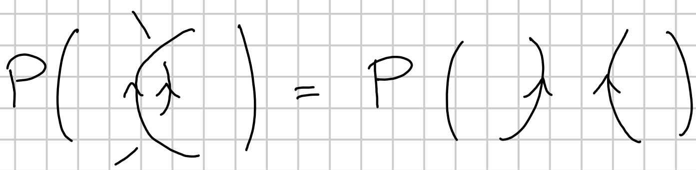
We need this to be equal also to

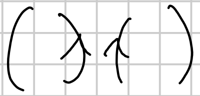


Because of (*), it suffices to prove one of the following:

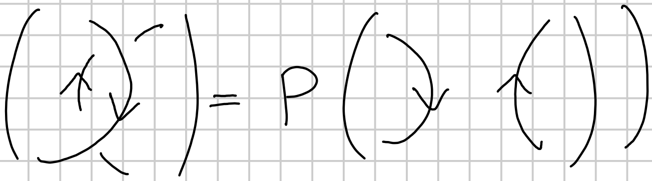


or

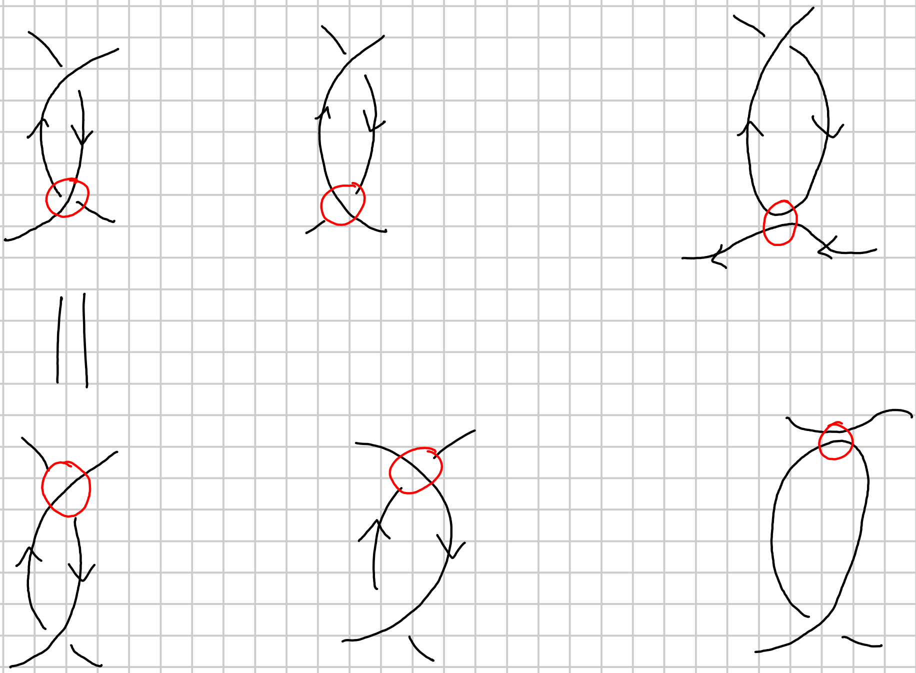


Now, exactly one of the two portions of the diagram described on the left is ascending. As before, for that diagram, the computation of P does not involve the crossings we are looking at, hence it coincides with the computation for $P($  $)$.

A similar argument works for

(i.e. P 

In fact:

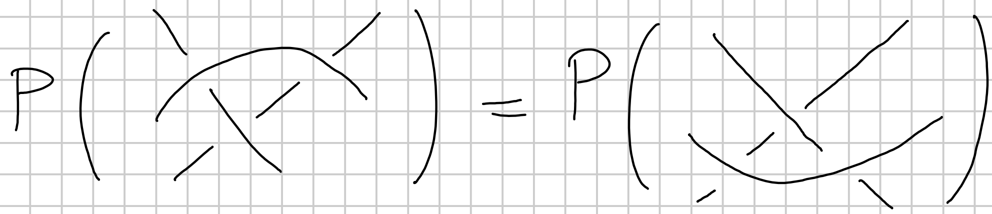


D_0^1

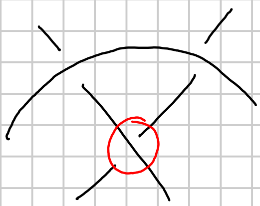
D_0^2

$P(D_0^1) = P(D_0^2)$ because I already proved invariance w.r.t. the I Reidemeister move.
Then I conclude as before.

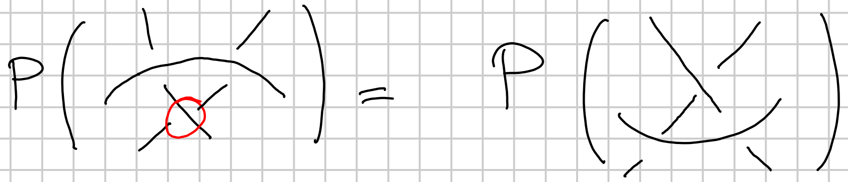
III Reidemeister move.



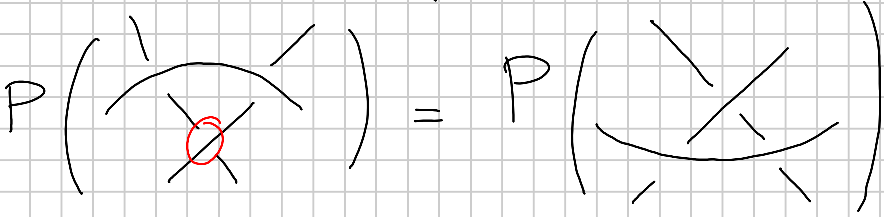
Again, one uses the skein formula to prove that the problem can be reduced to the case when the portion of the diagram on the left is ascending. In that case, also the portion on the right is ascending, and the equality follows from the Definition of P.



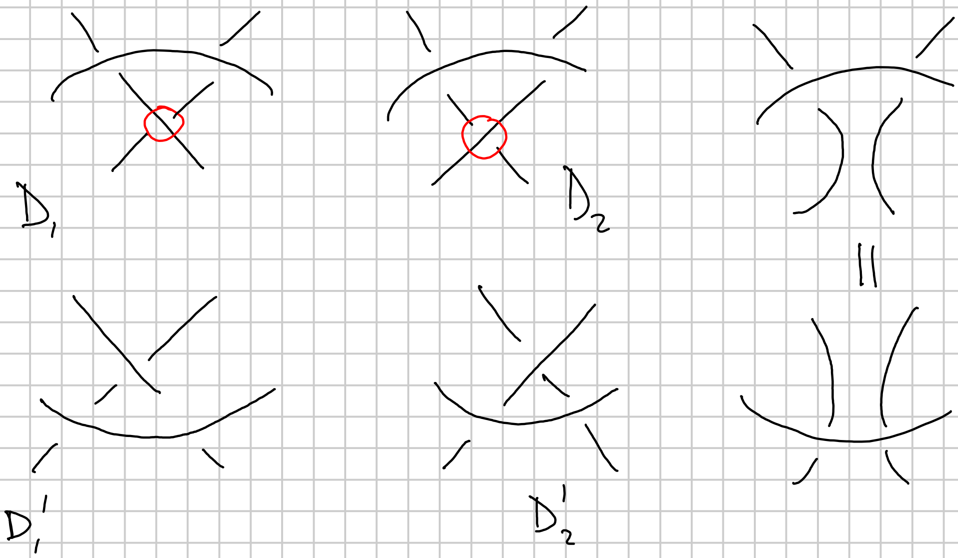
Suppose that this portion is not ascending, and that we need to change the red crossing in order to make it ascending. Then in order to conclude it is sufficient to show

 (a)

$\Updownarrow$

 (b)

If more crossing changes are needed, a similar strategy applies.
The only thing I need to prove is that these statements are equivalent (i.e. (a) $\Leftrightarrow$ (b)).



By the skein relation, $P(D_1) = P(D_1') \iff P(D_2) = P(D_2')$

i.e. (a) $\iff$ (b).

This proves invariance w.r.t. Reidemeister moves.

4 Invariance w.r.t. the choice of the ordering of link components.

5 $P(D) = \mu^{\#D-1}$ for EVERY ascending diagram.

(in fact 4 and 5 are proven at the same time).

After 4, 5 are proven, the HOMFLY polynomial is defined over $\mathcal{D}_n$ for every n, in a coherent way. Now, if L is any link, we choose a diagram D of L and put

$$P(L) = P(D).$$

If D' is another diagram, then $D \sim D'$

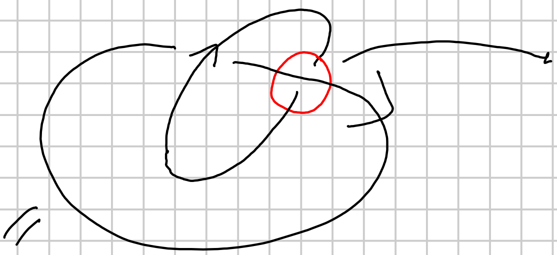
via Reidemeister moves within some D_n
(n = max number of crossings of the diagrams need to pass from D to D'). Hence

$$P(D)=P(D')$$

i.e. $P(L)$ is well-defined.

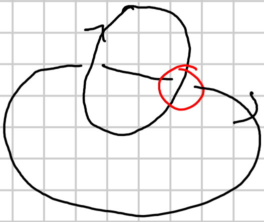
Examples

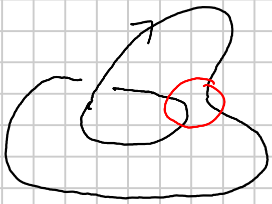
Hopf link (with negative lk number)



is a negative crossing

D_-

$D_+ =$  = unlink = ○ ○

$D_0 =$  = unknot = ○

$$\ell P(D_+)+\ell^{-1}P(D_-)+mP(D_0)=0$$

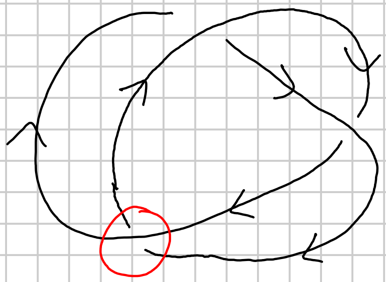
$$\ell P(\bigcirc\,\bigcirc)+\ell^{-1}P(\mathrm{Hopf})+mP(\bigcirc)=0$$

$$\ell\cdot\mu+\ell^{-1}x+m\cdot 1=0$$

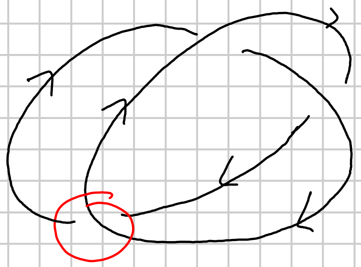
$$P(\text{Hopf}) = \frac{-m - l\mu}{l^{-1}} = \frac{-m + l\,\frac{l+l^{-1}}{m}}{l^{-1}} =$$

$$= l\left(-m + \frac{l^2+1}{m}\right) = \frac{-l m^2 + l^3 + l}{m}$$

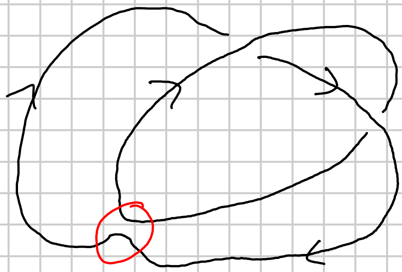
Trefoil



$D = D_-$



D_+ diagram of the unknot



D_0 diagram of the Hopf link

$$l\,P(D_+) + l^{-1}\,P(D_-) + m\,P(D_0) = 0$$

$$l\,P(\text{unknot}) + l^{-1}\,P(\text{trefoil}) + m\,P(\text{Hopf link}) = 0$$

$$l \cdot 1 + l^{-1} \cdot x + \cancel{m}\,\frac{-l m^2 + l^3 + l}{\cancel{m}} = 0$$

$$x = l\left(l m^2 - l^3 - 2l\right) = l^2\left(m^2 - l^2 - 2\right)$$

$P(\text{trefoil})$

Proposition: Let L be a link, and let $\overline{L}$ be mirror image of L. Then

$$P_L(\ell, m) = P_{\overline{L}}(\ell^{-1}, m)$$

If L is amphichiral (i.e. $L = \overline{L}$) then

$$P_L(\ell, m) = P_L(\ell^{-1}, m).$$

Proof.: P is completely determined by the skein formula. If we change all the crossings of a diagram D to obtain $\overline{D}$, in order to compute $P(\overline{D})$ we can just do the same computations as for D, with the roles of D_+, D_- switched in every application of the skein formula. Since

$$\ell P(D_+) + \ell^{-1} P(D_-) + m P(D_0) = 0,$$

this corresponds to switching ℓ with ℓ^{-1}.

Corollary: The trefoil is CHIRAL.

Theorem: The Conway normalization of the Alexander polynomial Δ_L satisfies:

$$\Delta_L(t) = P(L)\left(i, i\left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)\right),$$

$i^2 = -1$.

Proof: Define $f_L(t) = P(L)\left(i, i\left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)\right)$. Then $f_{\text{unknot}} = 1$ and from

$$\ell P(L_+) + \ell^{-1} P(L_-) + m P(L_0) = 0$$

by substituting $\ell = i$, $m = i\left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)$, we get

$$i f_L(t) + i^{-1} f_L(t) + i\left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) f_{L_0}(t) = 0,$$

i.e. $f_L(t) - f_L(t) + \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) f_{L_0}(t) = 0$,

which is the skein formula satisfied by the Alexander-Conway polynomial. The conclusion follows by uniqueness