Teoria dei nodi
9/4/19

- $f: M \to \mathbb{R}$ Morse if all crit. pts are non-deg
- every $f: M \to \mathbb{R}$ becomes Morse after small perturbation.

$n \geq 2$ $\quad$ $CAT \in \{TOP, DIFF, PL\}$

Jordan-Schönflies problem:

if $S \subset S^n$, $S \cong_{CAT} S^{n-1}$

$\overset{?}{\Rightarrow}$ $S^n = B_0 \cup B_1 \cup S$ with $\overline{B}_j \cong_{CAT} D^n$

$\partial B_j = S$.

Answer: YES $n = 2$ $CAT = TOP$

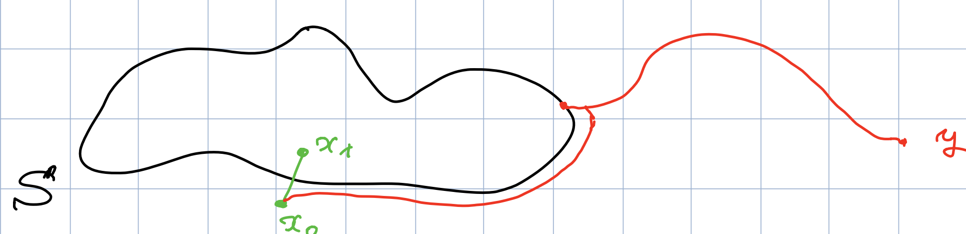
NO $n = 3$ $CAT = TOP$

(Alexander's horned sphere)

Thm: YES $n = 3$ $CAT = DIFF$

(YES $n = 3$ $CAT = PL$ similar proof)

Proof: fact $S^3 \setminus S$ has ≤ 2 components.

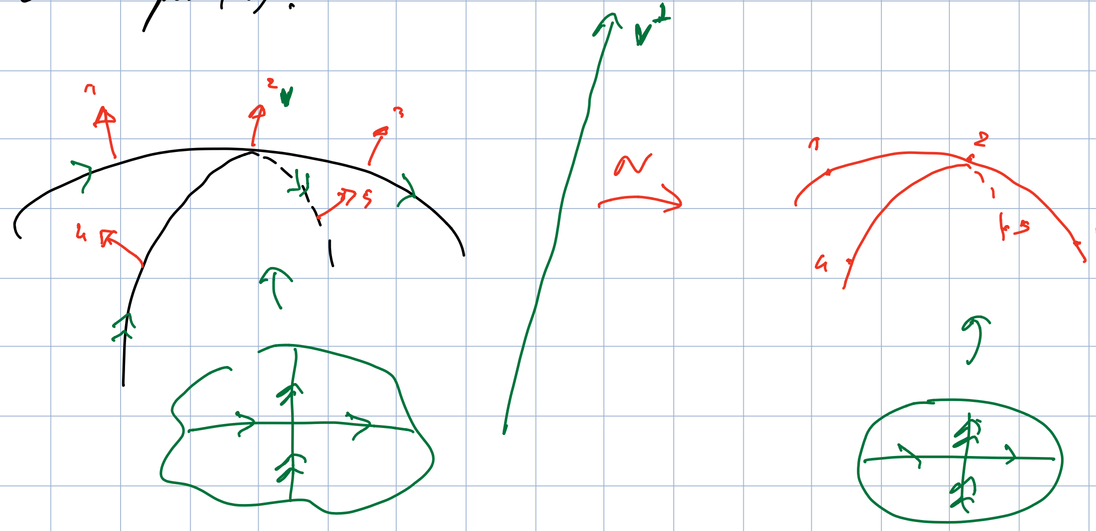


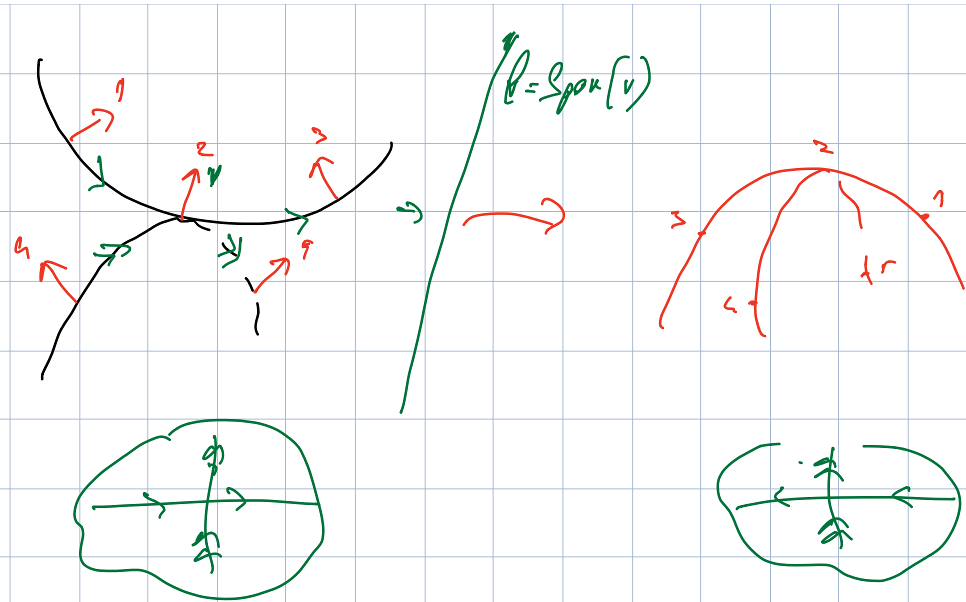
Claim: enough to show that $\Sigma \cong_{DIFF} S^2 \subset \mathbb{R}^3$ bounds B with $\overline{B} \cong D^3$. Enough because switching $x \in B$ with ∞ get other D^3.

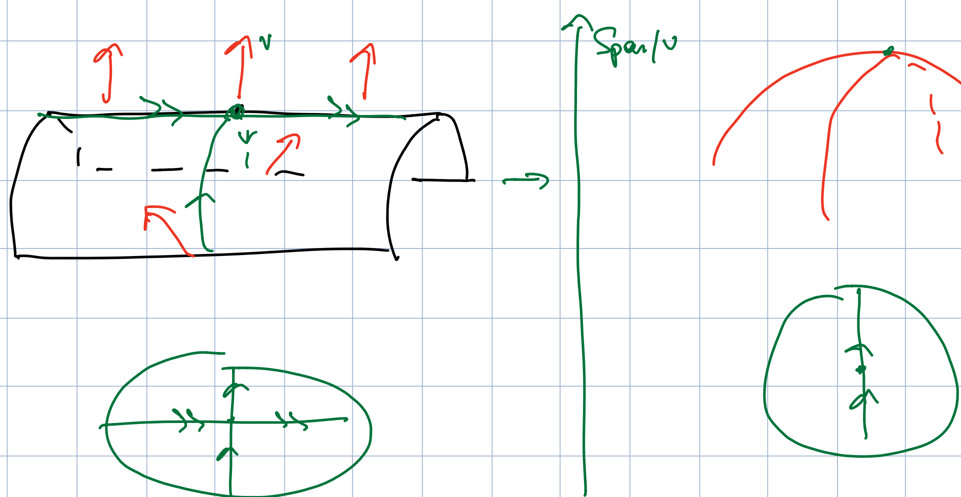
Claim: there exists a height function $f: \mathbb{R}^3 \to \mathbb{R}$ s.t. $f|_\Sigma$ Morse (height: $\perp$ projection on parametrized line; or simply f no critical pt & $f^{-1}(t) \cong \mathbb{R}^2$).

__Idea 1:__ take any $\perp$ projection and slightly perturb it so that restriction to Σ becomes Morse;

__Idea 2:__ take Gauss map $N: \Sigma \to S^2$ and choose $v \in S^2$ s.t. v & $-v$ are regular values of N; take $\ell = \perp$ projection on $\mathrm{Span}(v)$.





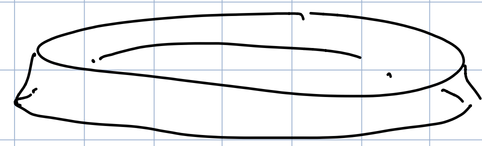


Have: $f : \mathbb{R}^3 \to \mathbb{R}$ height $h = f|_S$ Morse

wlog can assume that critical values $c_0 < c_1 < \dots < c_m$ all have one preimage. Choose $c_0 < a_1 < c_1 < \dots < a_m < c_m$.

notice that $h^{-1}(a_j)$ = collection of circles in $\mathbb{R}^2 \times \{a_j\} = f^{-1}(a_j)$;

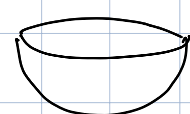
actually $h^{-1}([a_j-\varepsilon, a_j+\varepsilon])$ = collection of cylinders



Instead: $h^{-1}([c_j-\varepsilon, c_j+\varepsilon])$ = collection of cylinders

with 1 local exception:
signs (eigenvalues of Hh)

$+\ +$



$-\ -$

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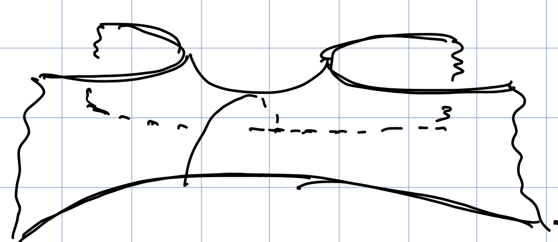
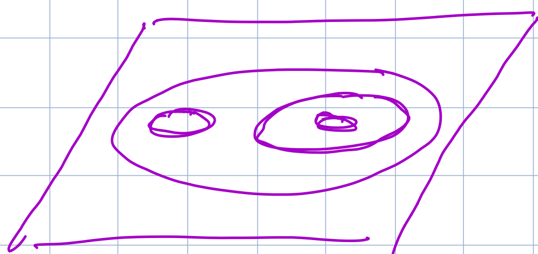
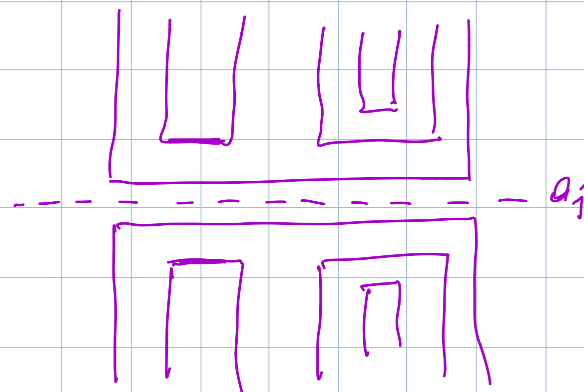


Perform surgery on S along each circle of $h^{-1}(a_j)$: individually

$\mathbb{R}^2 \times \{a_j\}$

If the circles in $h^{-1}(a_j)$ are nested I arrange the heights of surgeries so that the inner ones take place farther from $\mathbb{R}^2 \times \{a_j\}$:





Result: smooth closed surface s.t. each component consists of at most the following

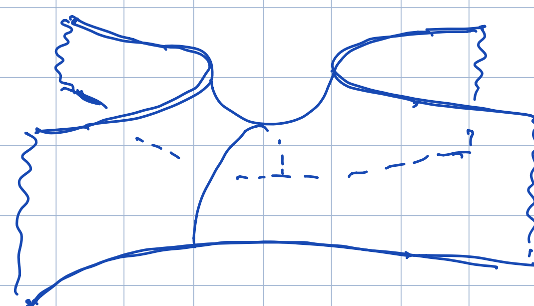
- one vertical tube on a Morse piece containing only one critical point
- some caps added doing surgery.

Claim: each component of the surgered sphere bounds B with $\overline{B} \cong_{DIFF} D^3$ ignoring other components. Reason:

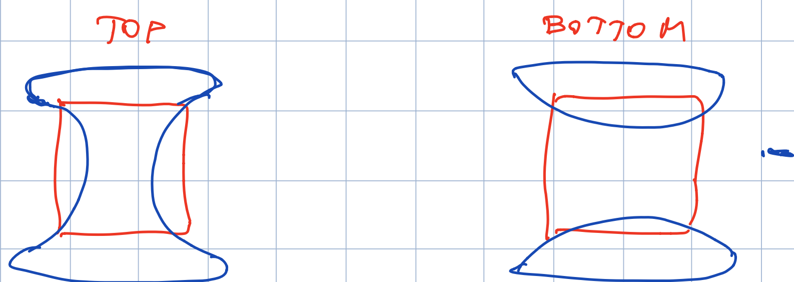
1) tube + caps

2) local min/max + caps

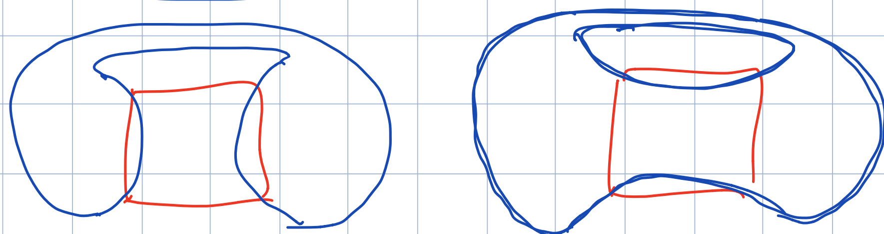
3) saddle + caps:



3.1

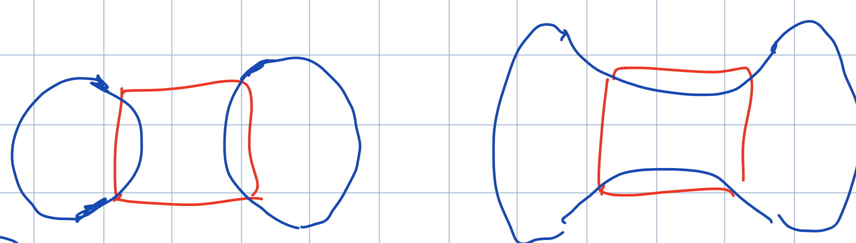


3.2



+ symmetric

3.3.

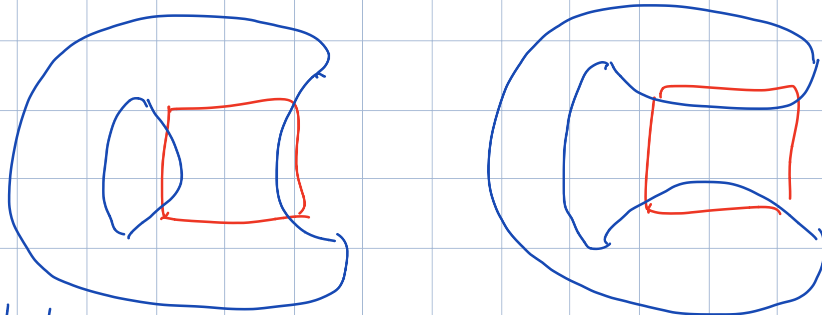


3.1 upside down

3.4

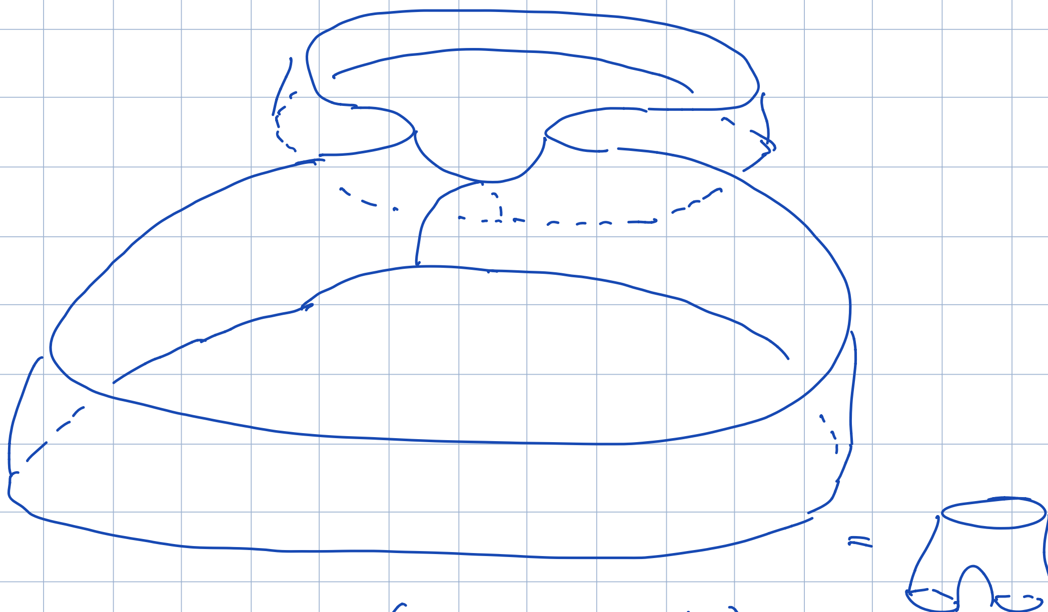
=

3.2 upside down



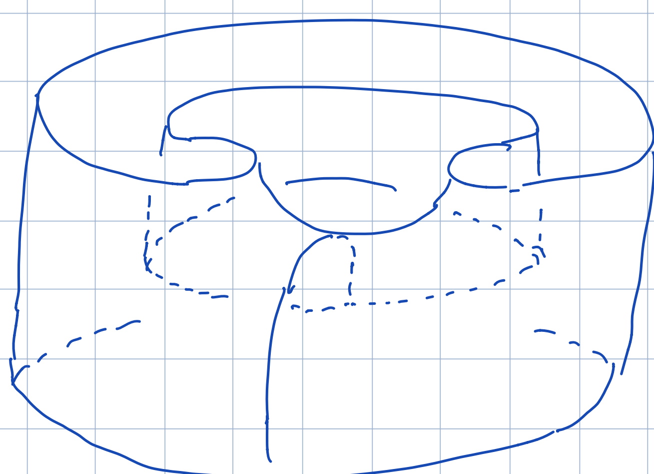
+ symmetric

3.1 :



+ 3 caps (2 bottom, 1 top)

3.2



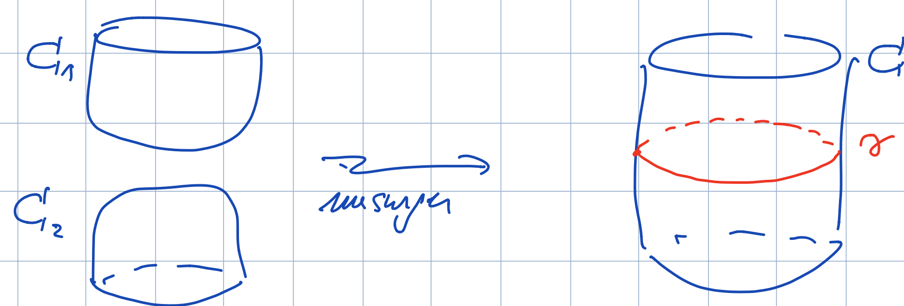
+ 3 caps at different heights

$\Longrightarrow$ OK does bound closed 3-disc.

Notice: never used so far the fact that $S \cong S^2$

Claim: the property that each individual component of the surface bounds a closed D^3 survives as I undo surgeries one at a time.

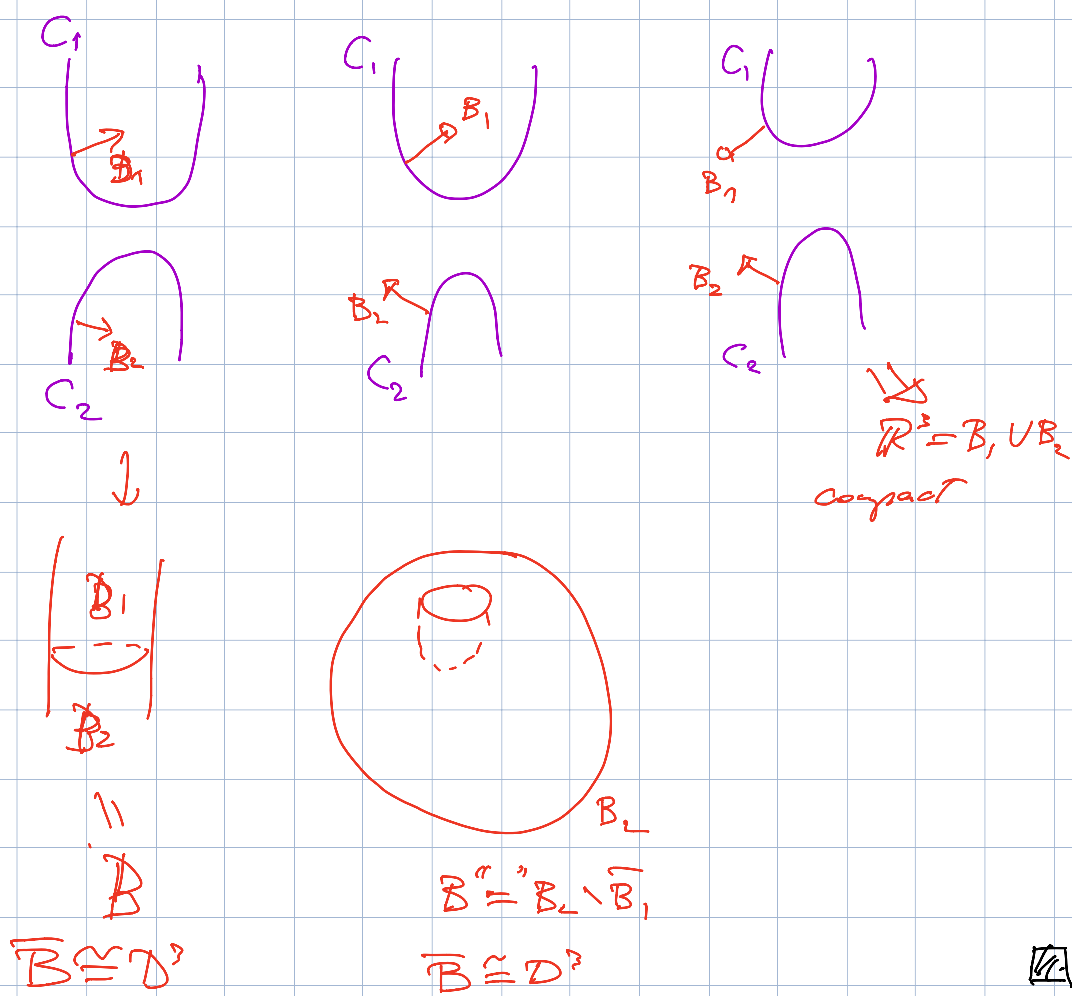
Remark:



If $C_1 = C_2$ then γ non-separating after undoing this surgery $\Longrightarrow$ remains non-separating after undoing later surgeries $\Longrightarrow$ non-separating on $S \cong S^2$: impossible by 2D-Schönflies (easy in DIFF category).

Now: $C_1 = \partial B_1$, $C_2 = \partial B_2$ $\overline{B}_i \cong D^3$
must show $C' = \partial B$ $\overline{B} \cong D^3$.

Three cases:

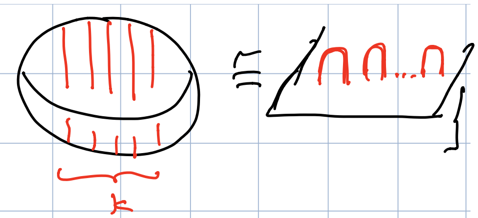


Bridge index:

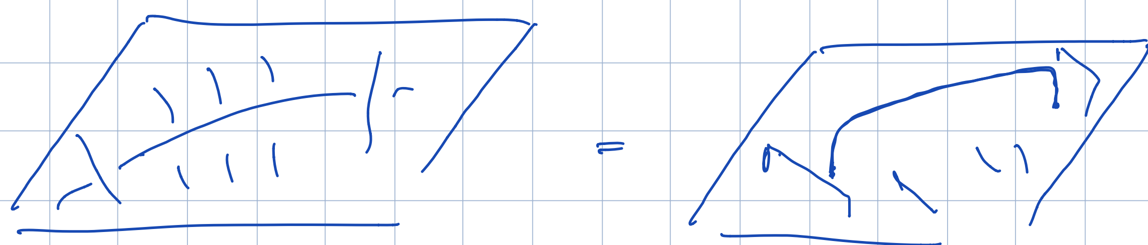
$L \subset S^3$ link

$$b(L) = \min\{ k : \ \exists\, S \underset{\text{DIFF}/\text{PL}}{\cong} S^2 \subset S^3 \ \text{ s.t.}$$

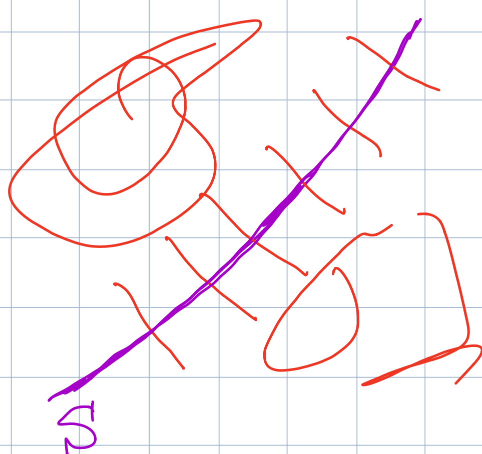
$S = \partial B_0 = \partial B_1$ and

$B_j, L \cap B_j \cong$ 

<u>Rem</u>: $b(L) \leq$ overarcs in a diagram

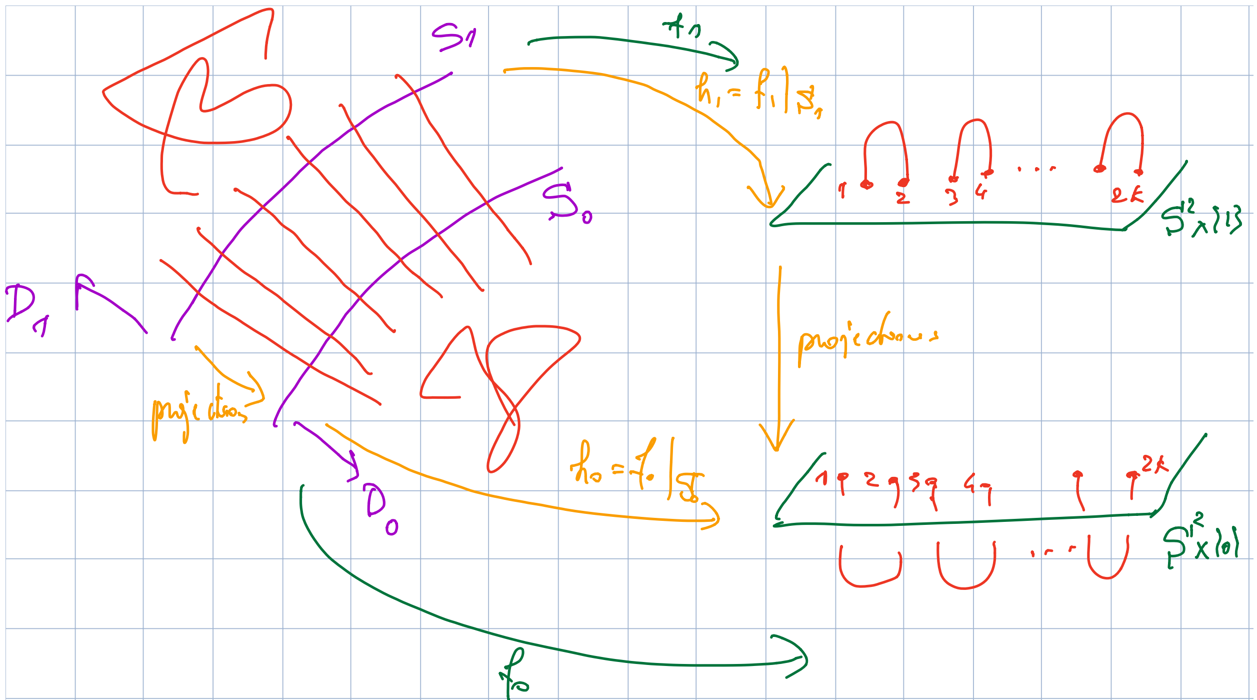


How to visualize bridge index k links:

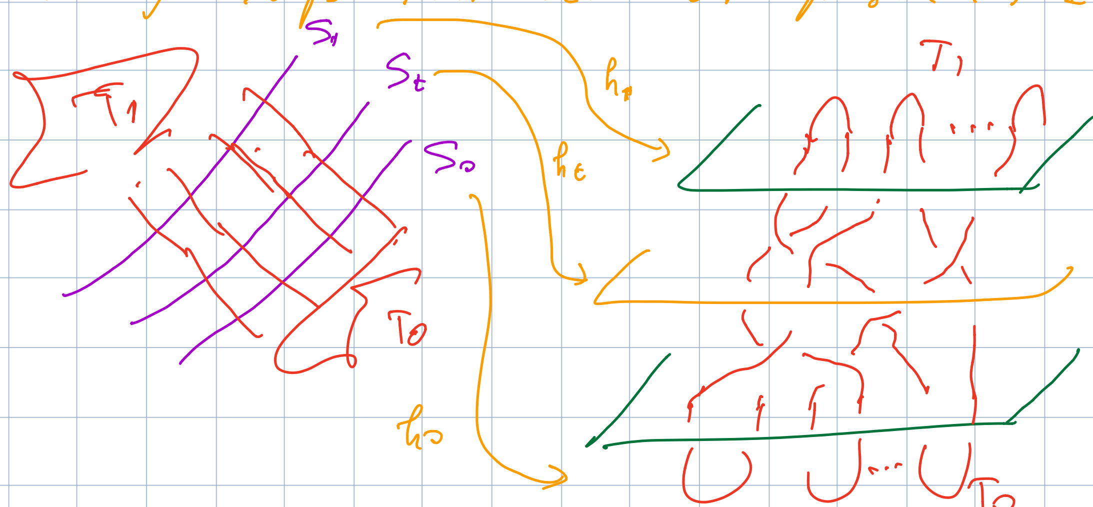


separately the two components of $S^3 \setminus S$ are trivial balls with tangle of k strands.

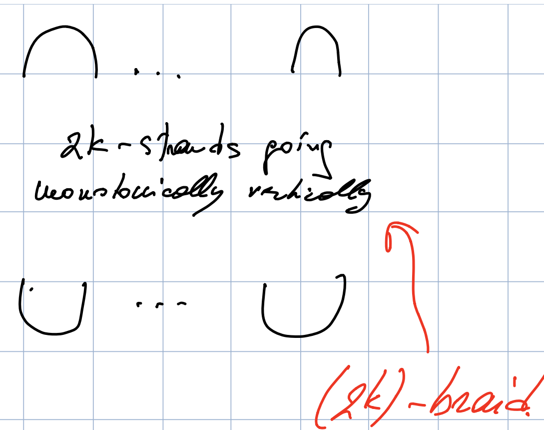
Take two parallel copies of S:



round composition of 4 maps gives $S^{1^2}\times\{0\} \circlearrowleft$ a loop orient. preserving mapping $\{1,2,\ldots,2k\}$ to themselves; it is therefore isotopic to id through maps that need not preserve $\{1,\ldots,2k\}$



bridge index k link is

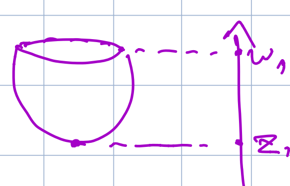


<u>Theorem</u>: $T \subset S^3$, $T \cong_{DIFF}$ torus, then $\exists K$ knot s.t. $T = \partial U(K)$.

<u>Proof</u>: <u>Claim 1</u>: there exists a sphere S s.t. $S \pitchfork T$ and some component of $S \cap T$ does not bound a disc on T.

<u>Proof of Claim 1</u>: take height function f s.t. $h = f|_T$ is Morse.

Suppose critical values are $z_1 < \dots < z_m$; take
$z_1 < w_1 < z_2 < \dots < z_m < w_m$
$T_j = h^{-1}((-\infty, w_j])$.

Notice $T_1 =$ 

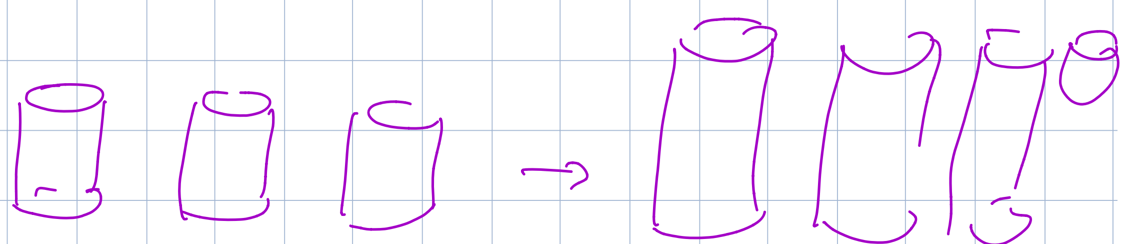
disc = 1-punctured sphere

Property of T_j : being union of punctured spheres

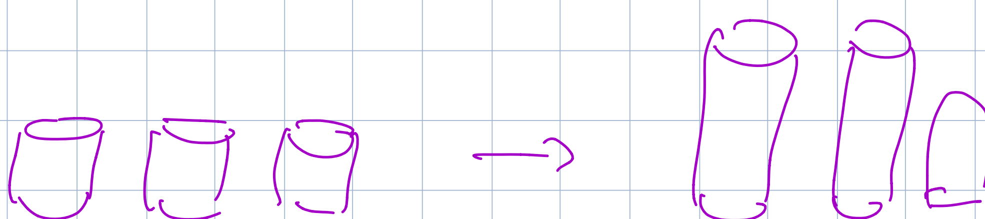
- true for $j=1$
- false for $j=n$ $\quad$ ($T_n = T$ torus)

Take smallest m s.t. T_m has property, T_{m+1} doesn't.

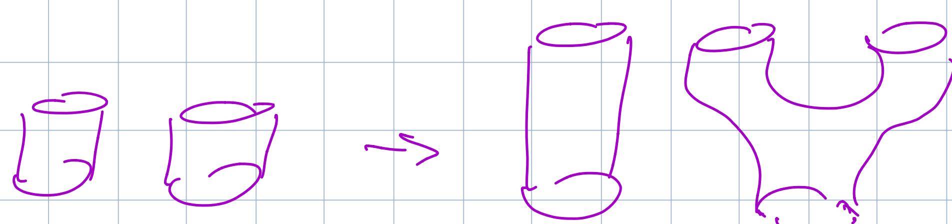
4 types of transitions depending on nature of critical value z_m :

1. 

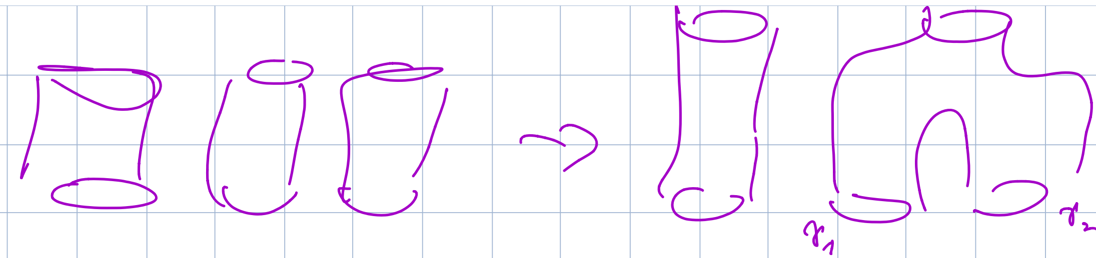
still holds

2. 

still holds

3. 

still holds

4. 

γ_1, γ_2 in different components
$\Rightarrow$ still holds.

$\Rightarrow$ transition is of type 4 with γ_1, γ_2 in same component.

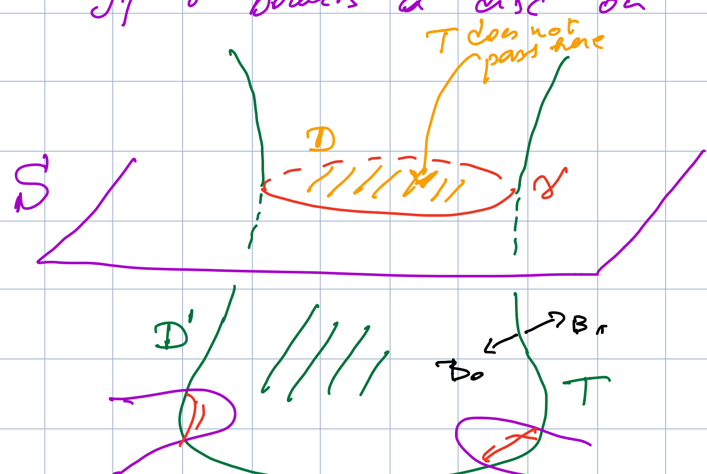
Hence γ_1 is non-sep. on T_{m+1}
$\Rightarrow$ remains non-sep in T.

Claim 1 proved with $S = h^{-1}(W_m)$

Claim 2 : conclusion

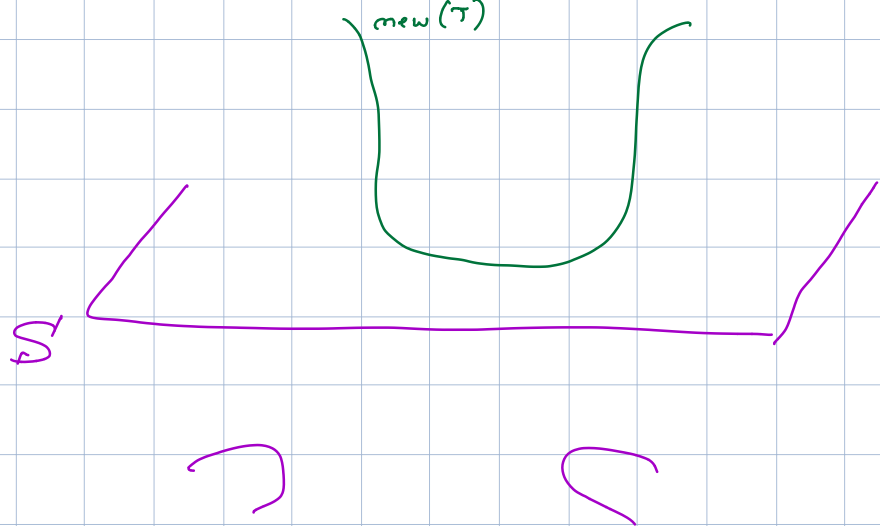
Take $\gamma \subset S \cap T$ innermost on S.

If γ bounds a disc on T:



$D \cup D' \cong S^2$
$\Rightarrow D \cup D' = \partial B_0$
$= \partial B_1$

can isotope T across B_0 removing γ in the intersection:

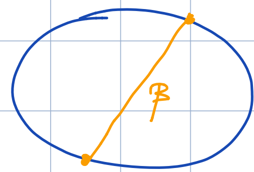


Continue until find γ innermost on S s.t. $\gamma = \partial\Delta$, $\Delta \subset S$, γ does not bound disc on T.

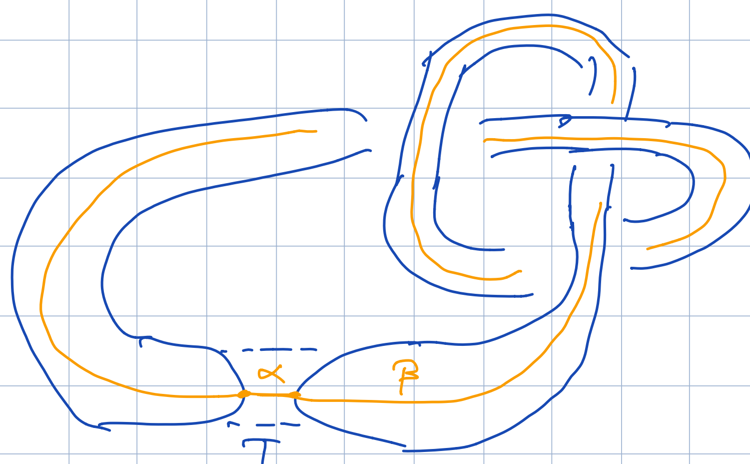
Surger T along Δ:

γ does not bound disc on T

$\Rightarrow$ Σ sphere $\Rightarrow$ bounds discs on both sides.

Take  B_0

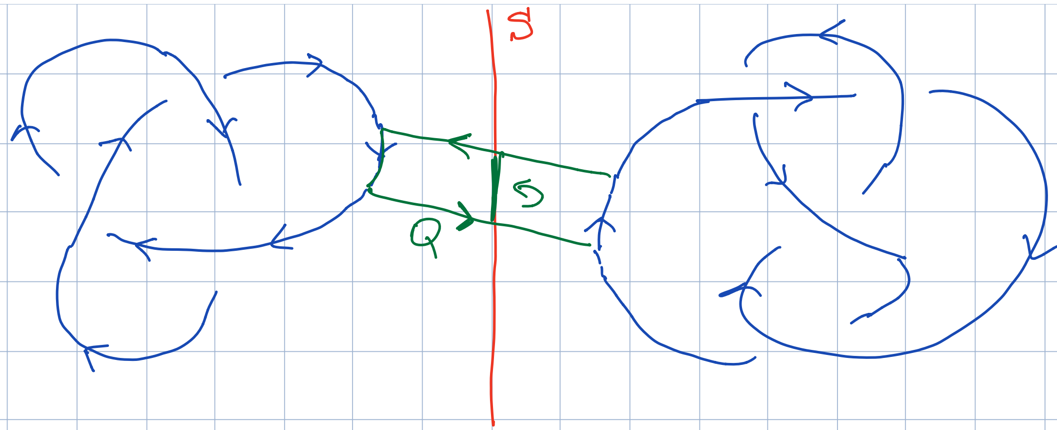
then $K = \alpha \cup \beta$ then $T = \partial U(K)$.

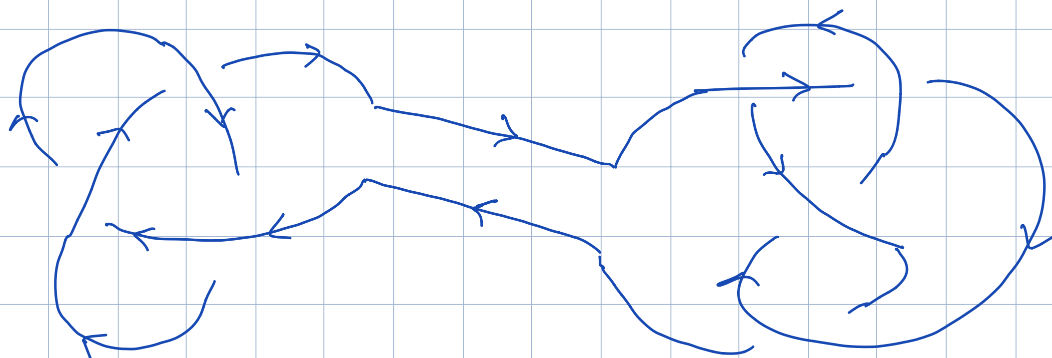


∎

<u>Def:</u> if $K_0, K_1 \subset S^3$ are oriented knots I define their <u>connected sum</u> $K_0 \# K_1$:

- Isotope K_0, K_1 s.t. $\exists\, S \cong_{DIFF} S^2$, $S = \partial B_0 = \partial B_1$ with $K_j \subset B_j$
- take $Q \cong [0,1] \times [0,1]$ oriented s.t. $Q \cap K_j = \{j\} \times [0,1]$ with orientation of ∂Q matching that of K_j and $Q \cap S = \{1/2\} \times [0,1]$
- $K_0 \# K_1 = K_0 \cup K_1 \setminus \left(\{0,1\} \times [0,1] \right) \cup \left(- \left([0,1] \times \{0,1\} \right) \right)$





Prop: $K_0 \# K_1$ is well-defined
independent of all choices made

For unoriented knots, $K_0 \# K_1$ is one of the (at most two) unoriented knots obtained as $\vec{K_0} \# \vec{K_1}$ using arbitrary orientations on K_0, K_1.

At most two: $(-K_0) \# (-K_1) = -(K_0 \# K_1)$

<u>Remark</u>: $K_0 \cong -K_0$ then $K_0 \# K_1$ well-def for unoriented.