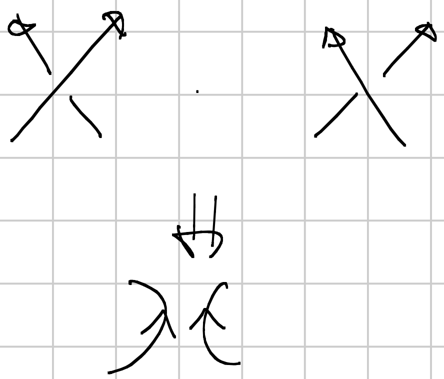
Genus of knots via the Alexander polynomial

Today: Every link L lies in S^3 (not $\mathbb{R}^3$)

Recap on the Seifert algorithm to construct Seifert surfaces:

From a diagram, one gets circles (to be filled with disks)

via



while every crossing gives rise to a band.

Hence, if $c = \#$ circles, $x = \#$ crossings, the resulting surface S deformation retracts onto a graph with c vertices and x edges.

Hence $\chi(S) = c - x$; if $g = \text{genus}(S)$,

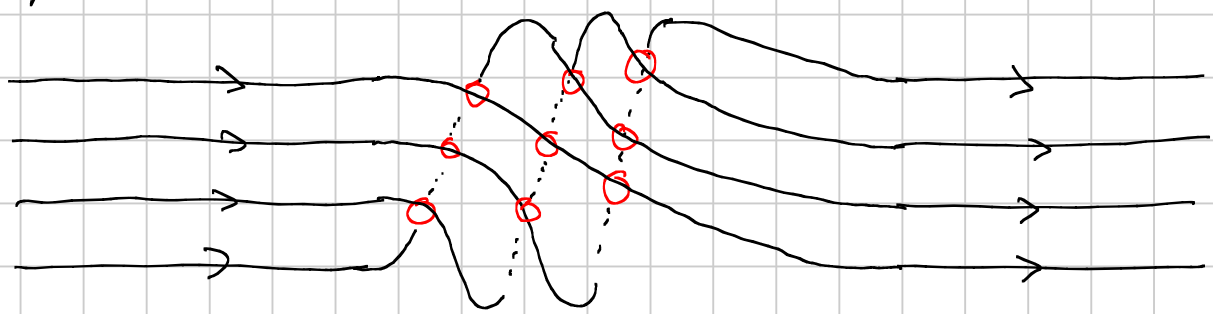
$\chi(S) = 2 - 2g - n$, $n = \#$ components of L

By comparing these formulae we get

$$g(S) = \frac{x - c - n}{2} + 1$$

Torus knots

$K_{4,3}$



At every crossing 

$\Longrightarrow$ 4 circles (p circles in general, for $K_{p,q}$)

crossings $= x = q(p-1)$ (in this case, $x = 9$)

Hence $g(S) = \dfrac{x - c - 1}{2} + 1 =$

$$= \frac{q(p-1) - p + 1}{2} = \frac{(p-1)(q-1)}{2}$$

I am assuming $GCD(p,q) = 1$

Corollary: $g(K_{p,q}) \leq \frac{(p-1)(q-1)}{2}$

Problem: How to find lower bounds?

A possible strategy is to look for invariants which can detect the genus.

Henceforth, we fix a link $L \subseteq S^3$ with n components, and set

$$E = E(L) = S^3 \setminus L \quad (\text{or } S^3 \setminus N(L))$$

There is a map (a homomorphism)

$$\pi_1(E) \xrightarrow{\varphi} \mathbb{Z}$$
$$[\gamma] \longmapsto \sum_{i=1}^{n} lk(\gamma, K_i)$$

when γ is itself a knot, and $L = K_1 \cup \dots \cup K_n$.

$\varphi =$ "total linking number".

Well-defined:

$$\begin{tikzcd} \pi_1(E) \arrow[r] \arrow[rrr, bend right, "lk(\cdot, K_i)"'] & H_1(E) \arrow[r] & H_1(S^3 \setminus K_i) \arrow[r, phantom, "\overset{\text{canonically}}{\cong}"] & \mathbb{Z} \end{tikzcd}$$

We now take the covering

$\tilde{E}_\infty \xrightarrow{p} E$ associated to $\mathrm{Ker}\ \varphi \triangleleft \pi_1(E)$

Covering theory $\Longrightarrow$ Since $\mathrm{Ker}\ \varphi$ is normal, this covering is regular, i.e.

$\mathrm{Aut}(\tilde{E}_\infty)$ acts transitively on fibers.

$$\mathrm{Aut}(\tilde{E}_\infty) = \frac{\pi_1(E)}{p_*(\pi_1(\tilde{E}_\infty))} \cong \mathbb{Z}$$

$(\varphi : \pi_1(E) \longrightarrow \mathbb{Z})$ is surjective

When L is a knot, $\tilde{E}_\infty$ is called the universal cyclic covering. We will call it the cyclic covering of E (it is an abuse).

$\mathrm{Aut}(\tilde{E}_\infty)$ has a preferred generator τ corresponding to $1 \in \mathbb{Z}$.

τ acts on $H_1(\tilde{E}_\infty; \mathbb{Z}) = H_1(\tilde{E}_\infty)$.

Thus we can make $\mathbb{Z}[t, t^{-1}]$ act on $H_1(\tilde{E}_\infty)$ as follows

$$\left(\sum_{i=-k}^{k} a_i t^i\right)([c]) = \sum_{i=-k}^{k} a_i (T_*)^i ([c])$$

$\mathbb{Z}[t, t^{-1}]$ = finite Laurent polynomials with integer coefficients

Definition: The Alexander module of L is $H_1(\tilde{E}_\infty)$, endowed with its $\mathbb{Z}[t,t^{-1}]$-module structure.

It is an invariant of L.

Two steps:

1. Understand the topology of $\tilde{E}_\infty$.
2. Develop some computable invariants of $\mathbb{Z}[t,t^{-1}]$-modules. (Elementary ideals of finitely presented modules).

Construction of $\tilde{E}_\infty$

Let S be a fixed Seifert surface for L. Since S is oriented,

$N(\mathring{S})$ = tubular neighbourhood of $\mathring{S} \cong$

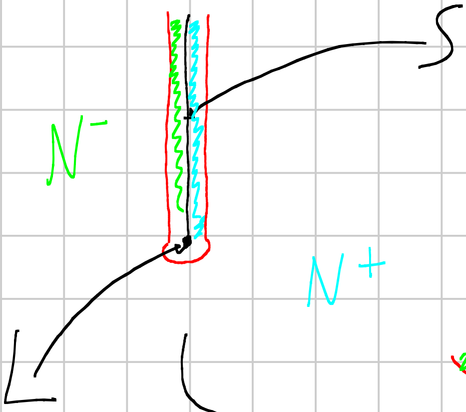
$$\cong \mathring{S} \times (-1,1)$$

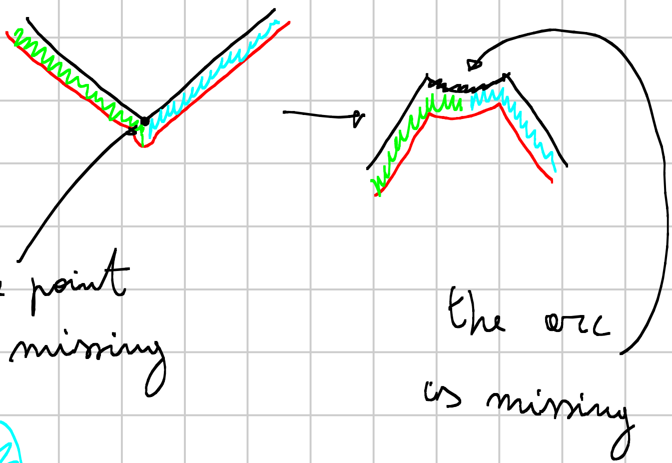
$N^+ = \mathring{S} \times (0,-1)$, $N^- = \mathring{S} \times (-1,0)$

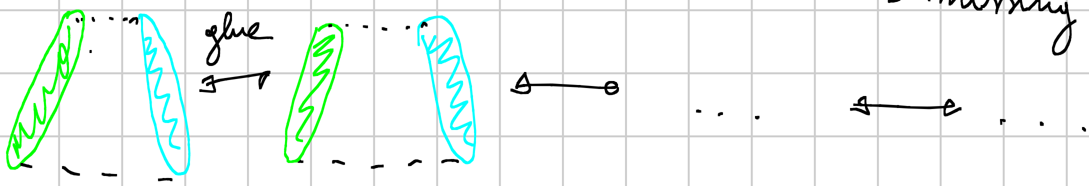
(well-defined because everything is oriented)

$Y = S^3 \setminus S$

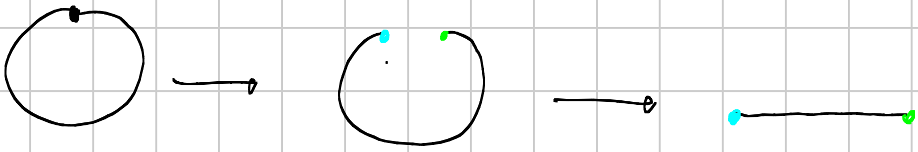
We would like to cut S^3 open along S

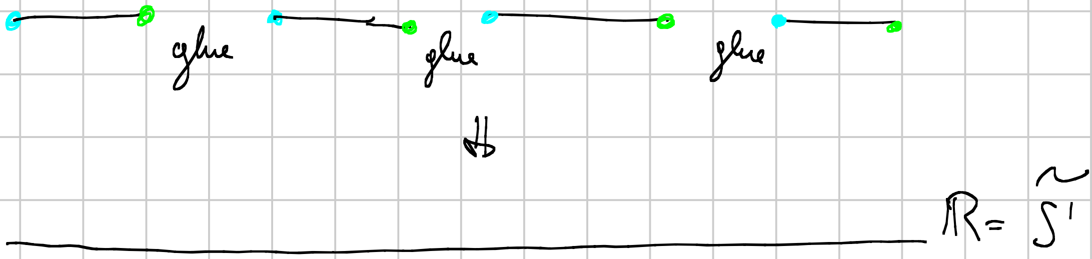




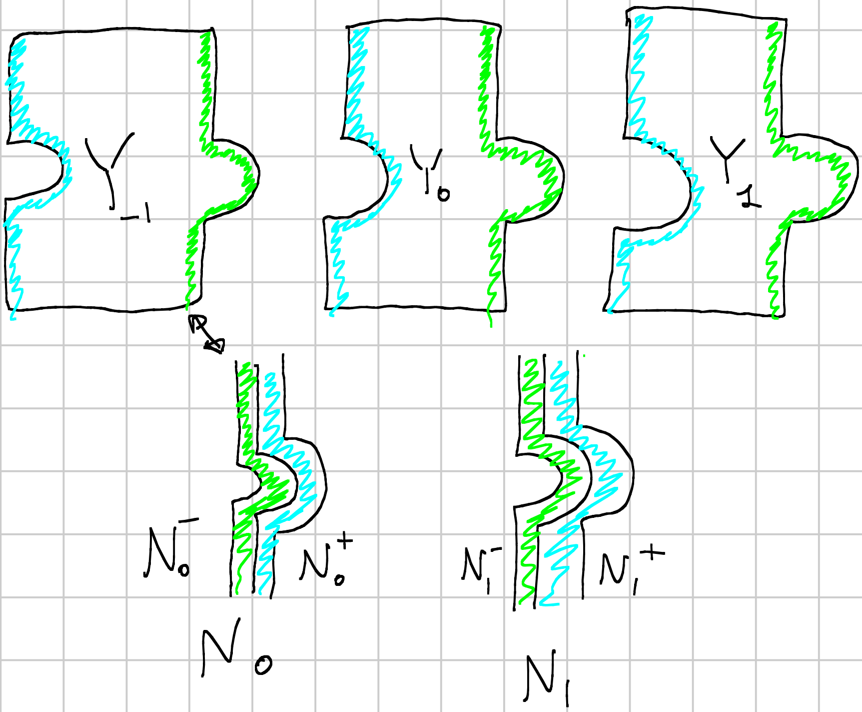


Analogy with S^1





Formally, take a countable number of copies Y_i, $i \in \mathbb{Z}$, of $Y = S^3 \setminus S$, and similarly N_i, $i \in \mathbb{Z}$



$\forall i \in \mathbb{Z}$, identify $N_i^- \subseteq N_i$ with $N_{i-1}^- \subseteq Y_{i-1}$

$N_i^+ \subseteq N_i$ with $N_i^+ \subseteq Y_i$

Let us call the resulting object $\tilde{E}_\infty$. We have an obvious projection $p: \tilde{E}_\infty \to E$ (send $Y_i \to Y$, $N_i \to N$ with the "identity"). By translating all the indices by 1, we get a covering automorphism τ which generates $\mathbb{Z} < \mathrm{Aut}(\tilde{E}_\infty)$ s.t. $\tilde{E}_\infty / \mathbb{Z} = E$,

hence $\mathrm{Aut}(\tilde{E}_\infty) = \langle \tau \rangle \cong \mathbb{Z}$

In order to conclude the proof that this $\tilde{E}_\infty$ is equal to the cyclic covering introduced above, we only need to check that every element of $\pi_1(E)$ with 0 total linking number lifts to a loop in $\tilde{E}_\infty$. But this follows from the construction and the description of linking numbers in terms of transverse intersections with S.

② How to study $H_1(\tilde{E}_\infty)$ as a $\mathbb{Z}[t,t^{-1}]$-module?

ELEMENTARY IDEALS OF FIN. PRESENTED MODULES.

Setting:
- R commutative ring with unity
- M is a fin. presented module over R, i.e. there exist free finitely generated R-modules F, E, and a short exact sequence

$$\cancel{0 \to} F \xrightarrow{A} E \to M \to 0$$

(i.e. a finite presentation).

Free module E with generators $e_1, \dots, e_k$
$\iff$ every $x \in E$ is of the form
$x = \sum_{i=1}^{k} a_i e_i$ for a <u>unique</u> choice of $a_i \in R$.

(in this case $E \cong R^k$).

After fixing free bases of F, E, the map A is a matrix with entries in R. It is called a PRESENTATION MATRIX for M.

Definition: A, A' matrices with R-entries. We define $\sim$ as the equivalence relation generated by:

1. permutation of rows or of columns;
2. adding an R-multiple of a row (column) to another row (column);
3. addition of some null columns;
4. $A \sim \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right)$.

Theorem: M R-module with presentation matrices A, A'. Then $A \sim A'$.

Conversely, if $A \sim A'$, then the modules presented by A and A' are isomorphic.

Proof: First statement: next week.
Second statement: easy exercise.
E.g. $A \rightsquigarrow \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right)$ means adding a generator and killing it.