Teoria dei nodi 5/3/19

Next lectures: Thu 7 – nothing
Tue 12 – me
Thu 14 – Frigerio.

338 5715369 – Send Whatsapp.

Seen: K knot $\Rightarrow H_1(E(K);\mathbb{Z}) = \mathbb{Z}_\mu$
$\Rightarrow \exists!\ \lambda_0$ longitude for K s.t. $[\lambda_0] = 0 \in H_1(E)$.

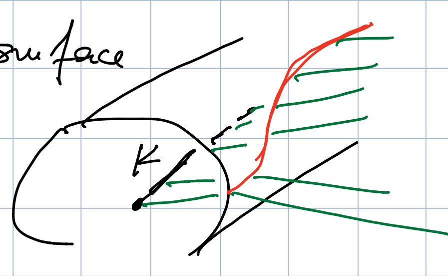
If λ any framing $\Rightarrow \lambda = \lambda_0 + p\mu \quad p \in \mathbb{Z}$.

Rem: p well-def: orient K
orient λ_0, λ accordingly
orient μ s.t.

if I reverse K all get reversed $\Rightarrow p$ remains same.

$[\lambda_0] = 0 \in H_1(E) \Rightarrow \lambda_0$ bounds in $E(K)$
a (singular) oriented surface

$\Rightarrow K$ also does



Can show via H_* that K bounds an embedded surface.

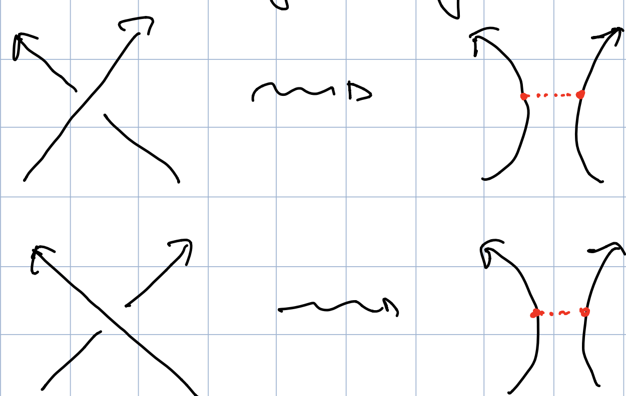
direct proof

Prop: $\vec{L}$ oriented link $\Rightarrow \exists\, \Sigma \subset S^3$ embedded oriented surface s.t. $\partial \Sigma = \vec{L}$ respecting orientation.

Σ = Seifert surface

Proof (Seifert algorithm):

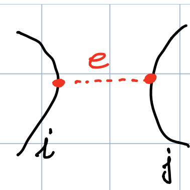
1. Remove crossings using orientation



2. Have bunch of circles in $\mathbb{R}^2$;
   declare level 0 those innermost
   declare level 1 not level 0 but encircles only level 0
   ...

3. For every circle at some level i, choose horizontal disc at height i bounded by circle and oriented ...

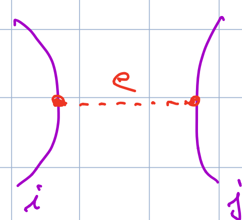
4. Claim: if



(nothing to see for $i=j$)

then $ex(i,j)$ meets no disc.

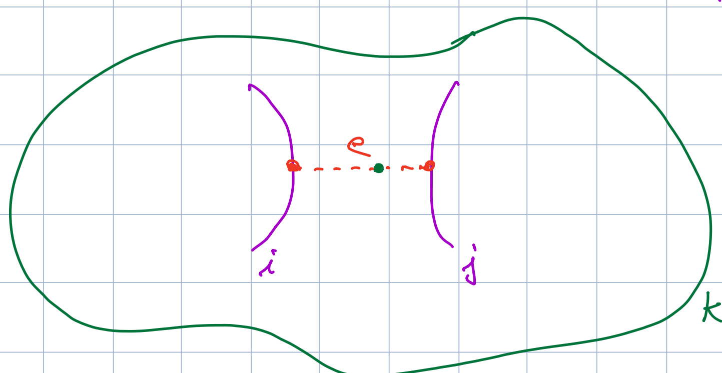
By contradiction



$i<j$

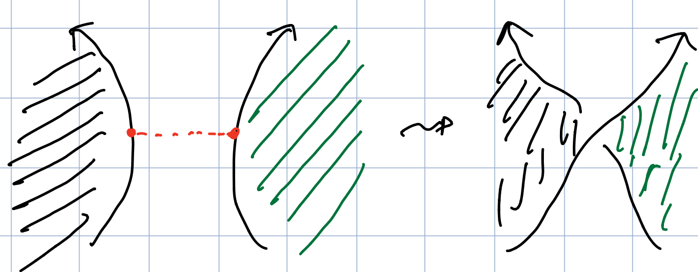
$ex(i,j) \cap$ disc at height

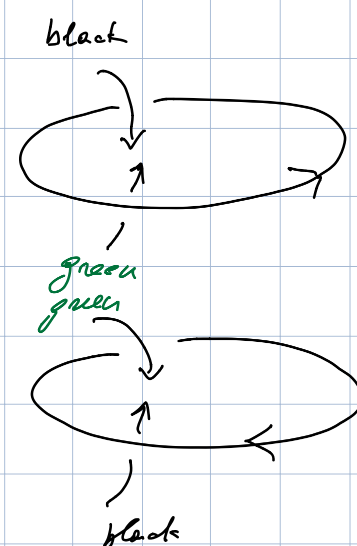
$\Longrightarrow i<k<j$

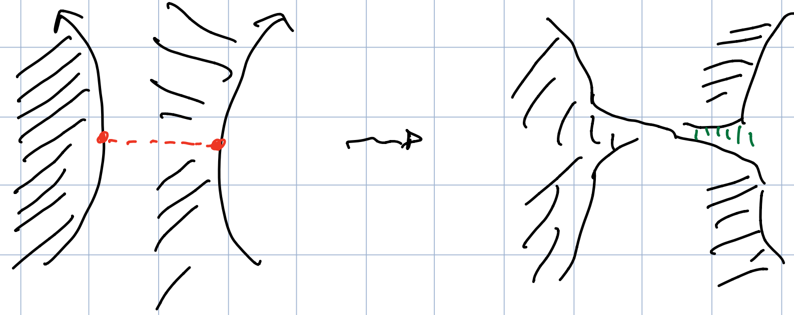


$\Longrightarrow$ circle of level k encircles one at level j

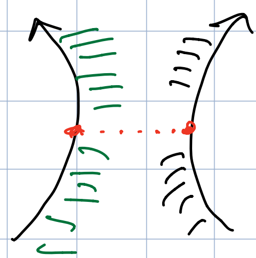
5. Restore crossings:







+ symmetric picture

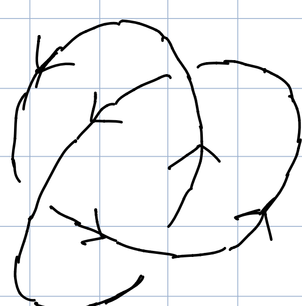


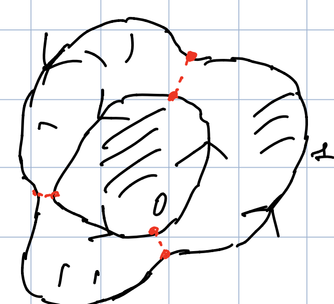
cannot happen.

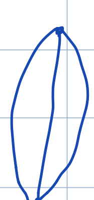
∎

Remark: if L link, $K \subset L$, Σ Seifert surface for L $\Rightarrow$ Σ determines a longitude for K that may not be its preferred longitude.

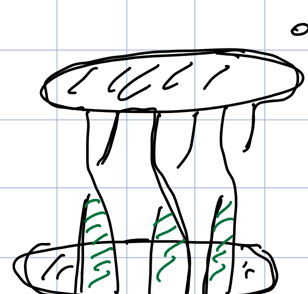
Examples:

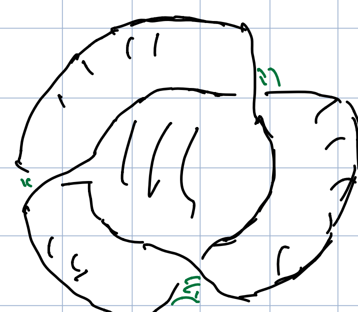






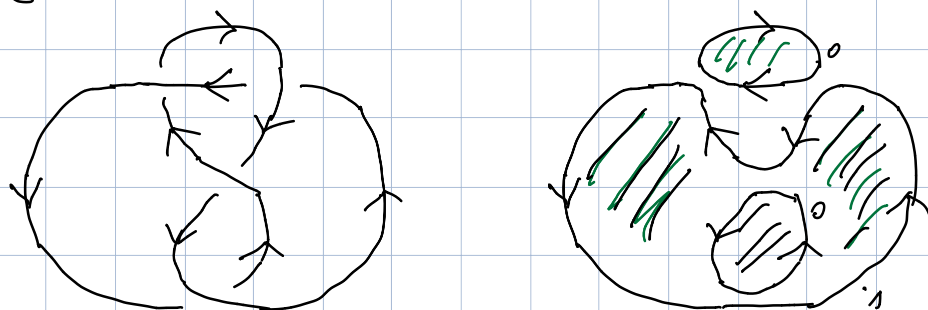
$\simeq$





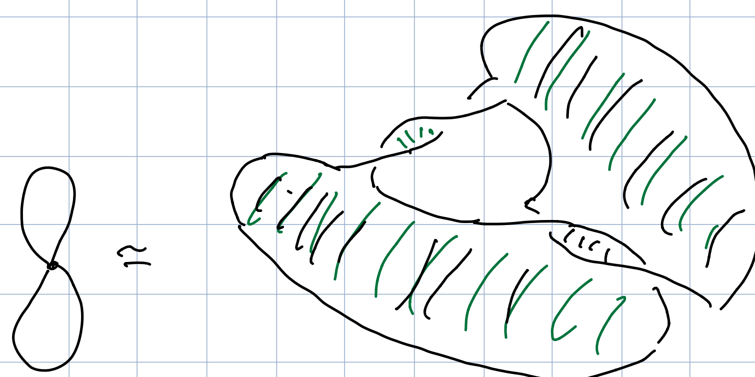
$\chi(\Sigma) = -1 \Rightarrow g = 1$ $\qquad$ $\chi = \underbrace{0}_{\chi(\hat{\Sigma})} - \underbrace{1}_{\#2}$

Figure-8

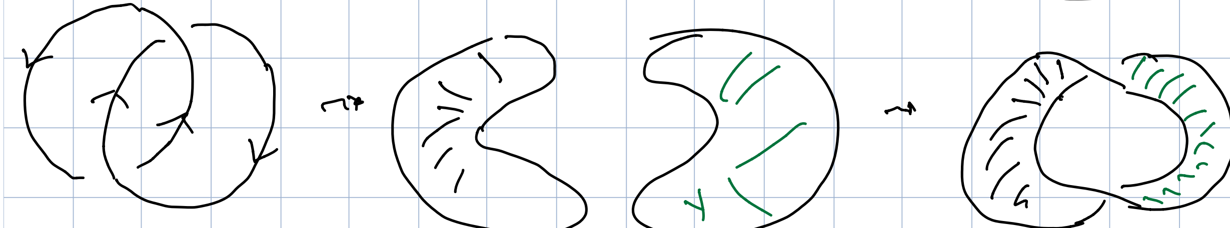


$\chi = -1$

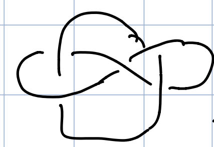
$g = 1$



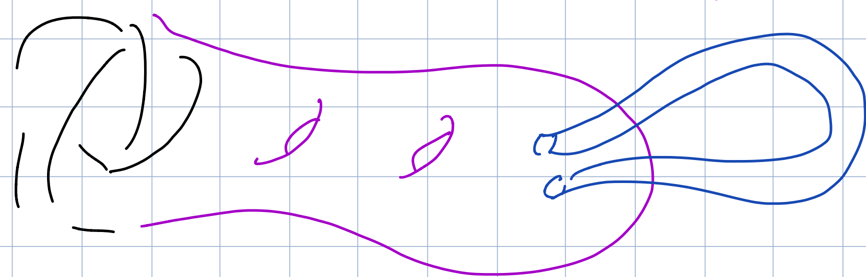
Hopf link:



$g = 0$; framing induced on each compo is not the natural 0 one.

<u>Exercise</u>: compute $g(\Sigma($  $))$

$\uparrow$ from Seifert algo.



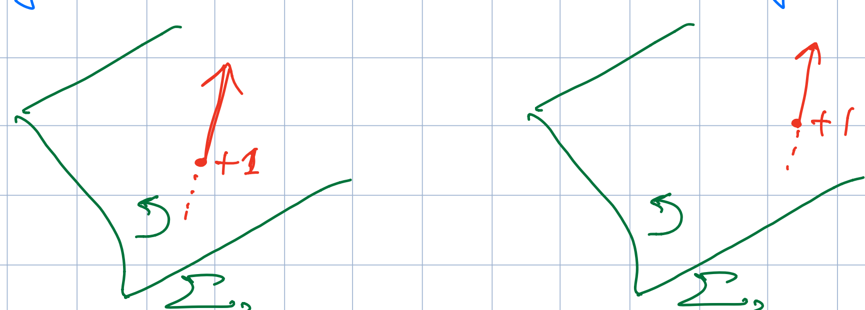
$g(\vec{L}) = \min\{ g(\Sigma) : \Sigma \text{ Seifert for } \vec{L} \}$

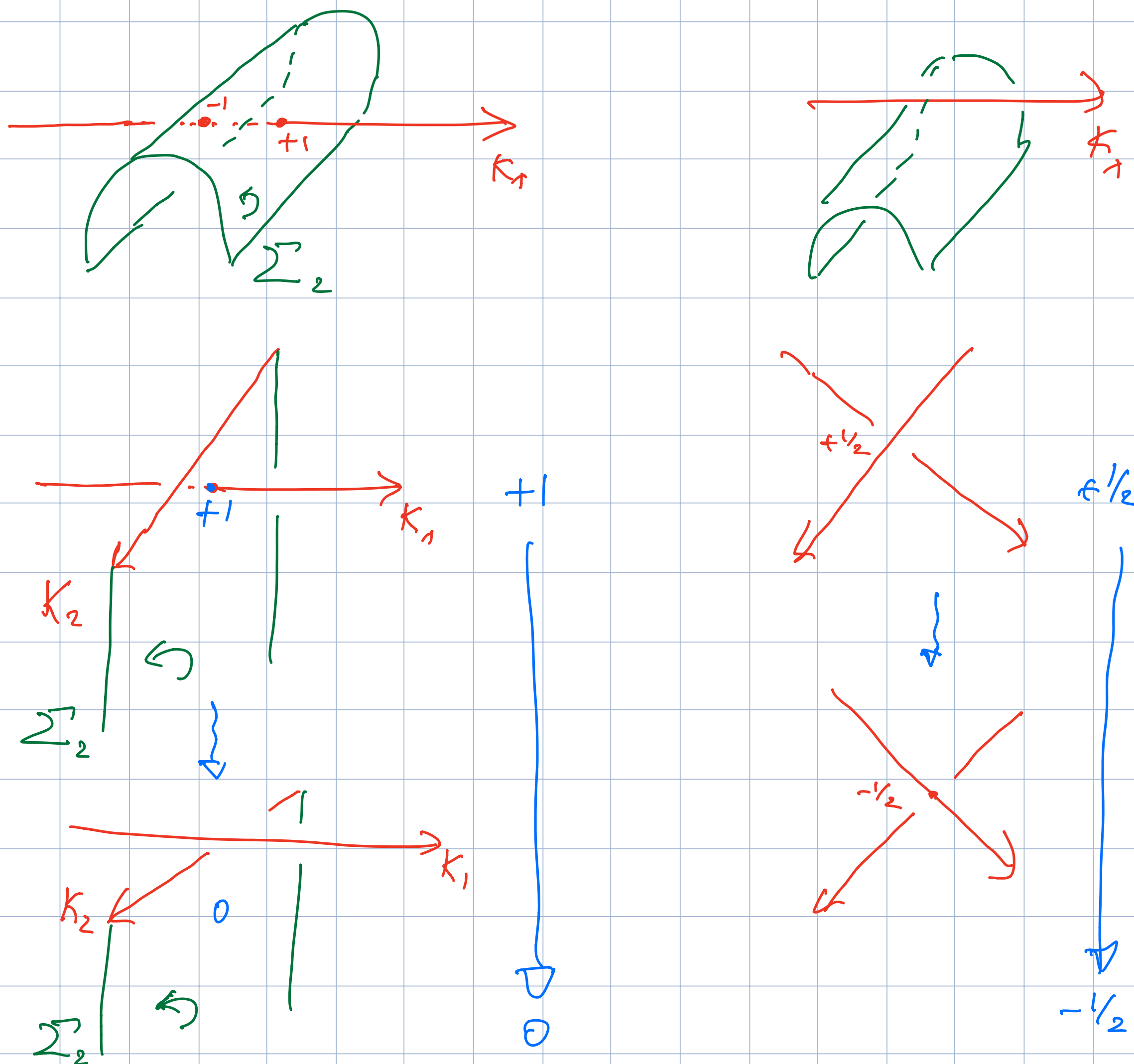
<u>Remark</u>: $g(K) = 0 \iff K$ trivial.

<u>Prop.</u> $lk(\vec{K}_1, \vec{K}_2) = \#_{alg}(K_1 \pitchfork \Sigma(K_2))$

$\Sigma(K_2)$ any Seifert surf. for K_2

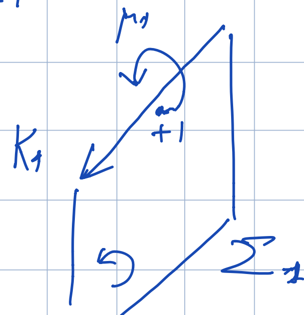
<u>Proof</u>: if I isotope $\vec{K}_1$ away $\Sigma(K_2)$ both 0.
Enough to see that difference is unchanged along isotopy.





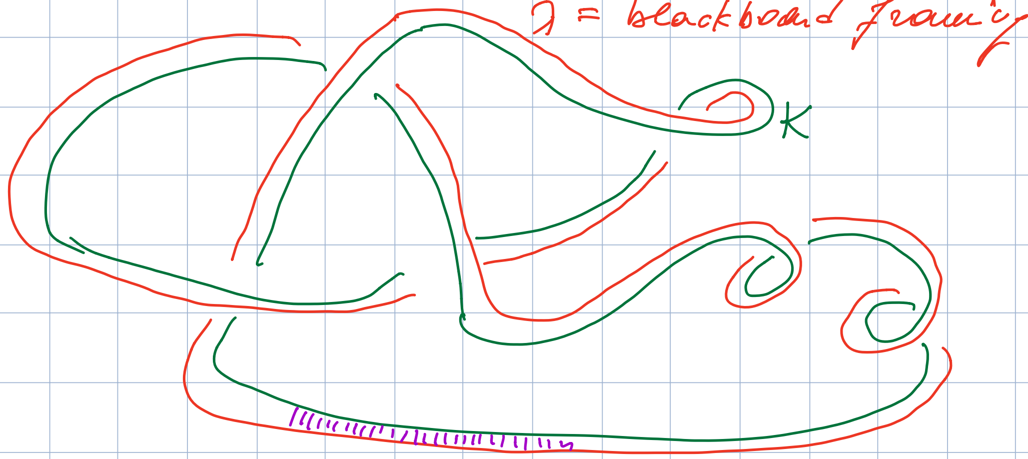
Cor: $lk(\vec{K}_1, \vec{K}_2) = p$ if $[\vec{K}_2] = p \cdot [\vec{\mu}_1] \in H_1(E(K_1))$

Proof: $\#_{alg}(\mu_1, \Sigma(K_1)) = +1$



moreover $\#_{alg}(\gamma, \Sigma_1)$ depends on $[\gamma] \in H_1(E(K_1))$. $\square$

Def: diagrammatic (blackboard framing):



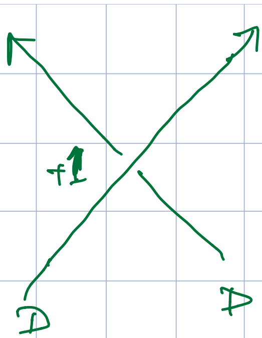
Def: D <u>knot</u> diagram; self-linking number of D

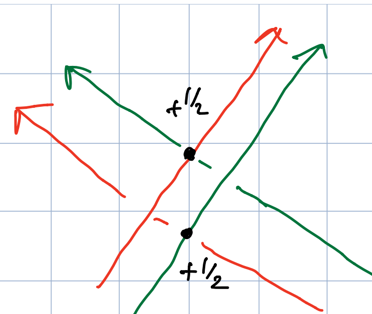
$$\mathrm{slk}(D) = \sum \begin{cases} +1 \\ -1 \end{cases}$$

(indep. of global orientation).

Prop: the integer representing the blackboard framing of D is $\mathrm{slk}(D)$.

Proof: claim: $\mathrm{slk}(D) = \mathrm{lk}(D, \lambda)$ $\quad$ λ = blackboard longitude

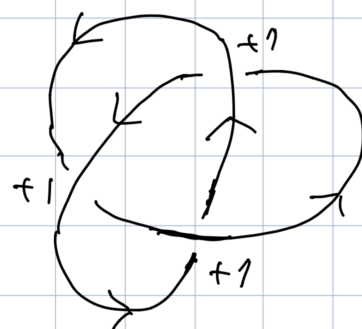




$lk(K,\lambda) = p$ if $[\lambda] = p \cdot [\mu]$ in which case λ is represented by $p \in \mathbb{Z}$. ▨

Rem: $lk(K, \lambda_0) = 0$

Example:

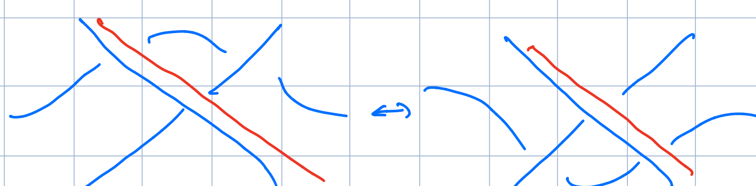


$slk = +3$

$\Rightarrow$ blackboard framing is not the preferred one.

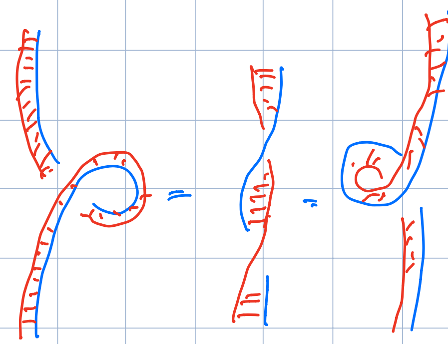
Framed Reidemeister moves (blackboard framing)

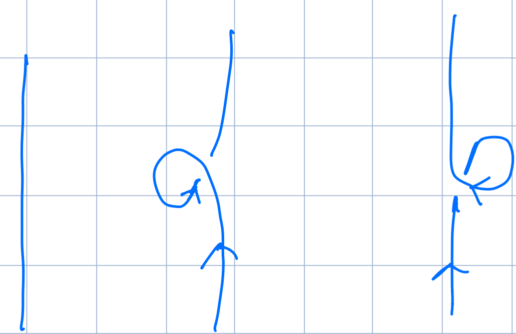
Easy: R_{II}



preserve blackboard framing.

slk = 0   slk = +1   slk = +1

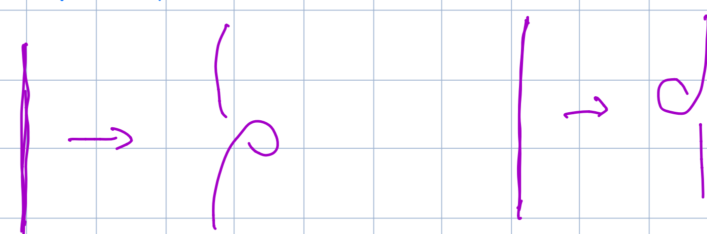




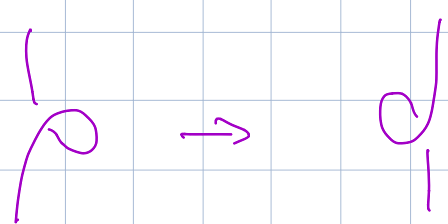
slk = 0   slk = −1   slk = −1

similar picture

Rem: we said



are "the same" for unframed diagrams because they become same after a 180° rotation. But:



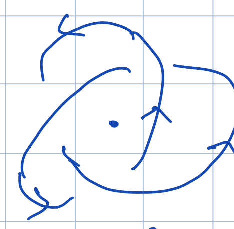
even if it preserves the framing is not generated by RII/III.

In fact: if α parametrizes K (unoriented)

$$\beta = \pi_{\mathbb{R}^2} \circ \alpha \implies \frac{\beta'}{\|\beta'\|} : S^1 \to S^1$$

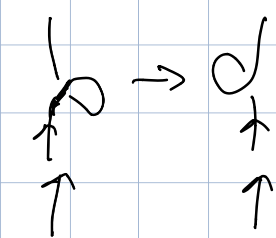
$\left|\deg\left(\frac{\gamma'}{\|\gamma'\|}\right)\right|$ well-def & invariant under planar isotopy + R_{II}/R_{III}.

$\left|\deg \frac{\gamma'}{\|\gamma'\|}\right|$ = rotation number



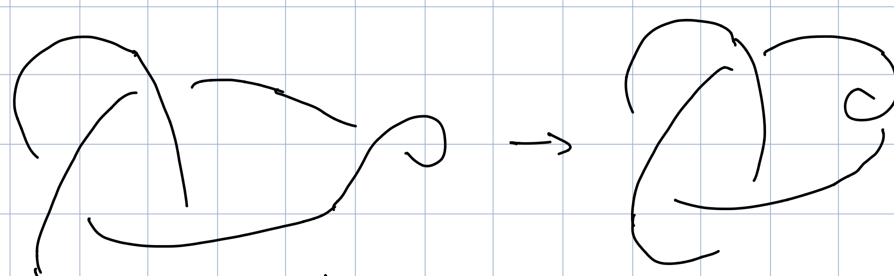
+2

But it does change under



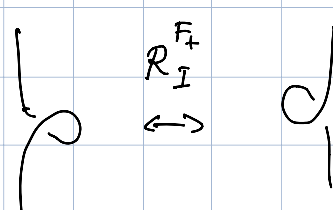
−1 +1

contrib to rot. number

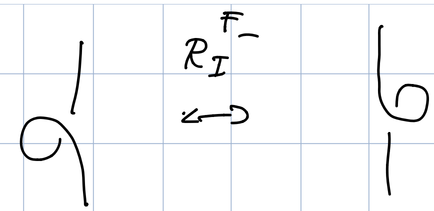


1 3

Two framed R_I moves:

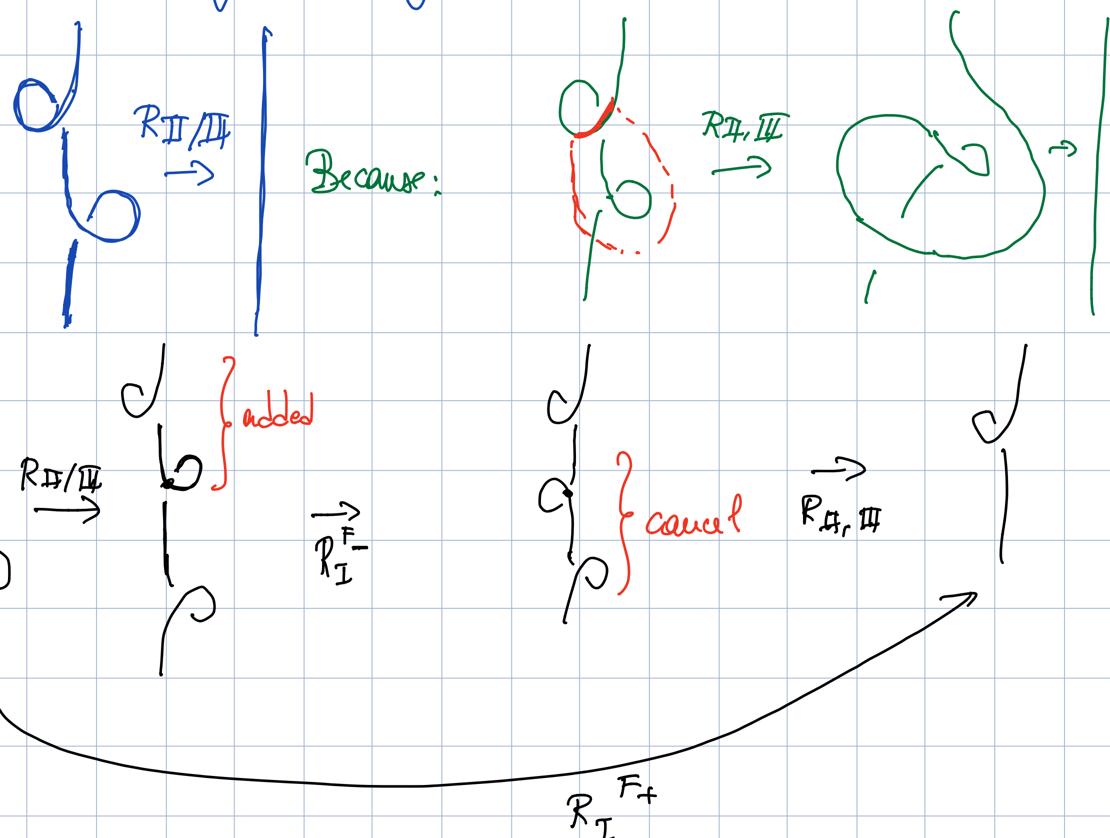


$R_I^{F_+}$



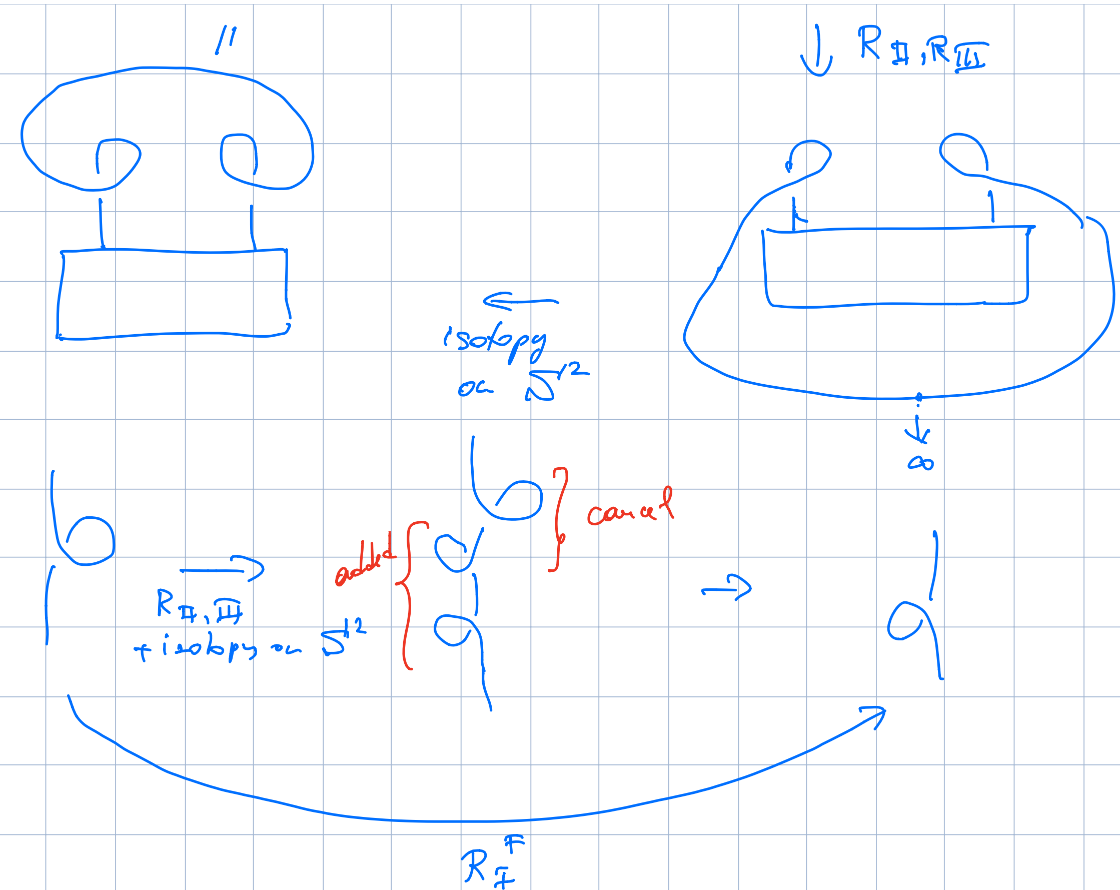
Rem: $R_I^{F_\pm}$ generated by $R_I^{F_\mp}$, R_{II}, R_{III}. In fact:

Chain



Prop:
- R_I^F generated by $R_{II,III}$ + isotopy on S^2
- isotopy on S^2 generated by R_I^F, R_{II}, R_{III}.

Proof:

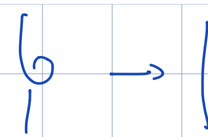


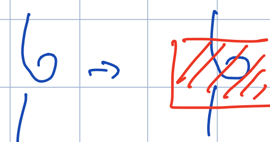
other implication similar. $\boxed{\%}$

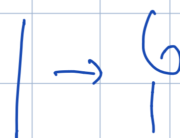
Thm: D_1, D_2 represent the same framed link via blackboard framing iff related by R_I^F, R_{II}, R_{III}
iff related by R_{II}, R_{III}, isotopy on S^2.

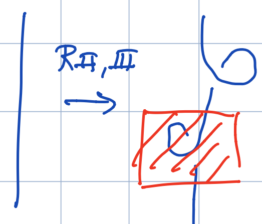
Proof: "if" obvious because R_I^F, R_{II}, R_{III} preserve framing.

"only if": D_1, D_2 represent isotopic links
$\Rightarrow$ $\exists$ sequence of R_I, R_{II}, R_{III} relating them.
I modify the sequence as follows:

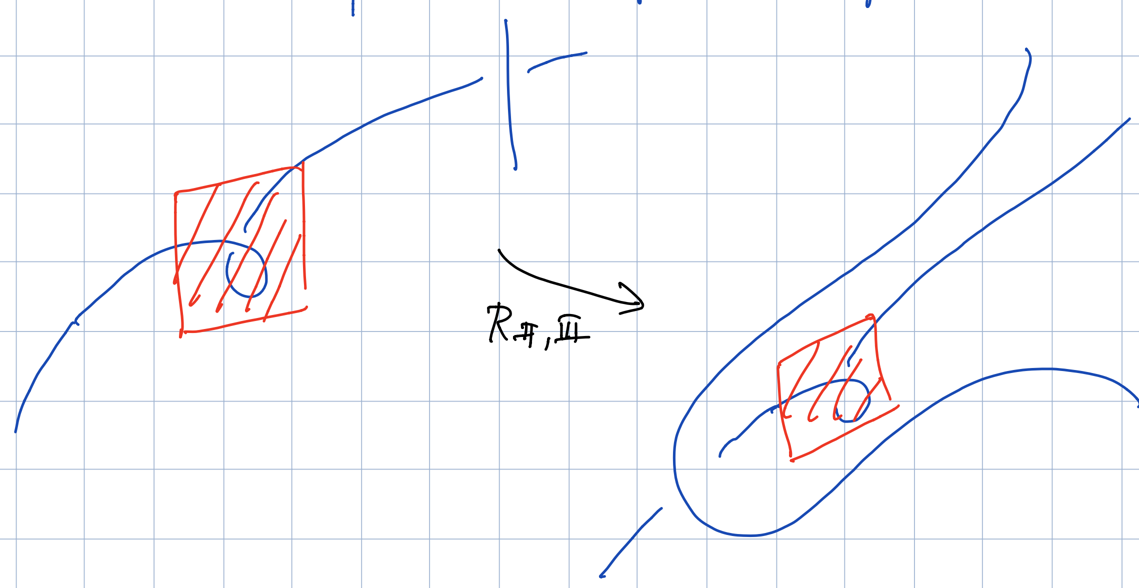
if I have a move R_I:  I don't do it

keep the curl frozen in tiny box 

if I have a move R_I:  I perform instead



I can continue the sequence using $R_{II,III}$ except that some planar isotopy now requires $R_{II,III}$:

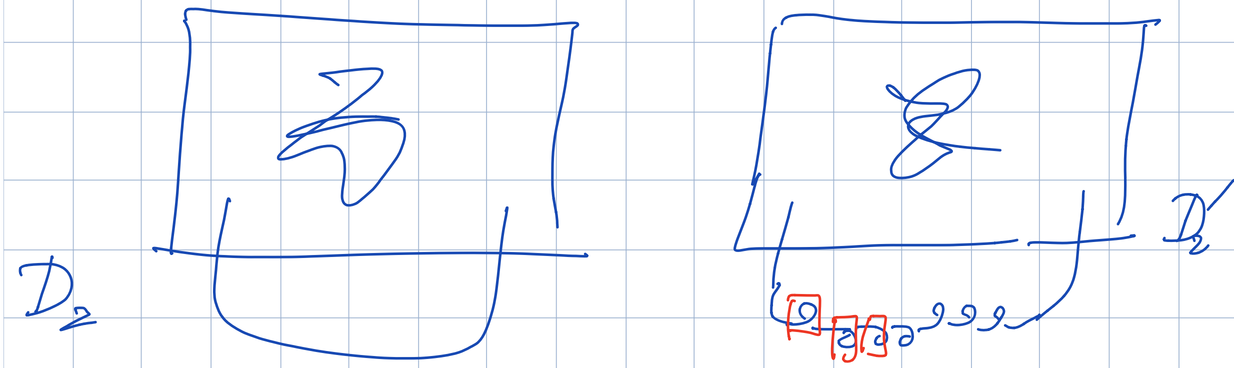


In the end have diagram D_2' that

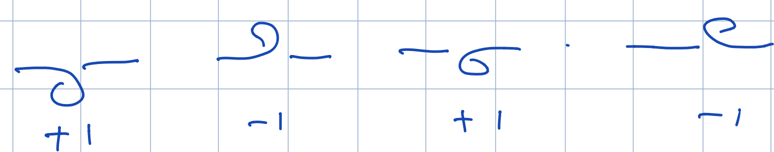
- coincides with D_2 except for some 
- only used R_{II}, R_{III} $\Rightarrow$ D_2' framed isotopic to D_1

$\Rightarrow D_2'$ framed isotopic to D_2

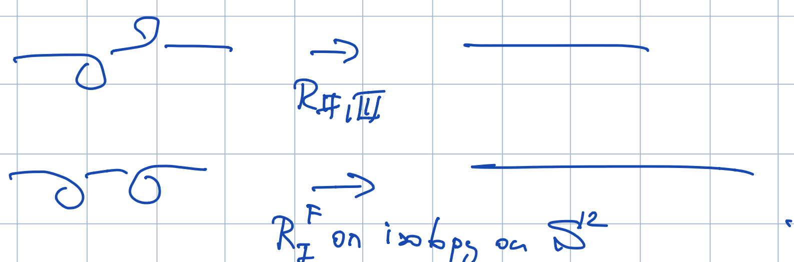
Wlog can assume all curls are consecutive



4 types:



algebraically these curls cancel; so they also cancel



∎