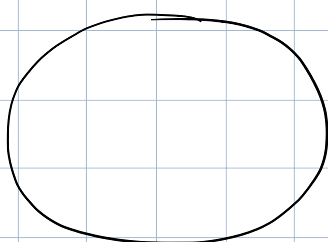
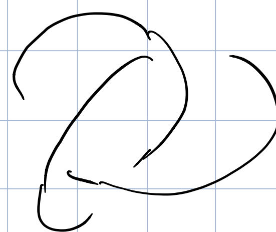
Teoria dei nodi 26/2/19

still pdf blackboard + animated blackboard video
→ www



unknot



trefoil

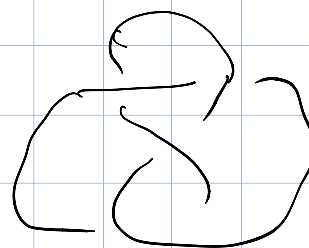
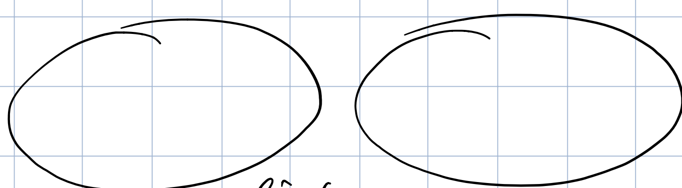
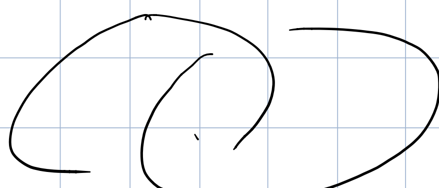


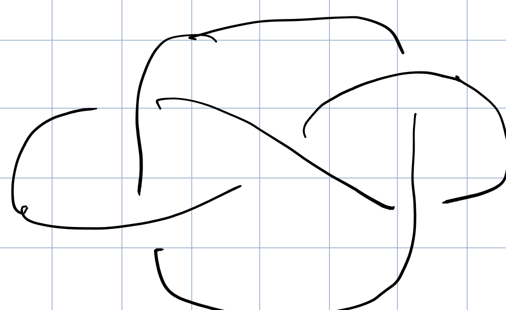
figure-eight



2-unlink



Hopf link



Whitehead link

Naif def: image of an embedding of S^1 in $\mathbb{R}^3$ or S^3

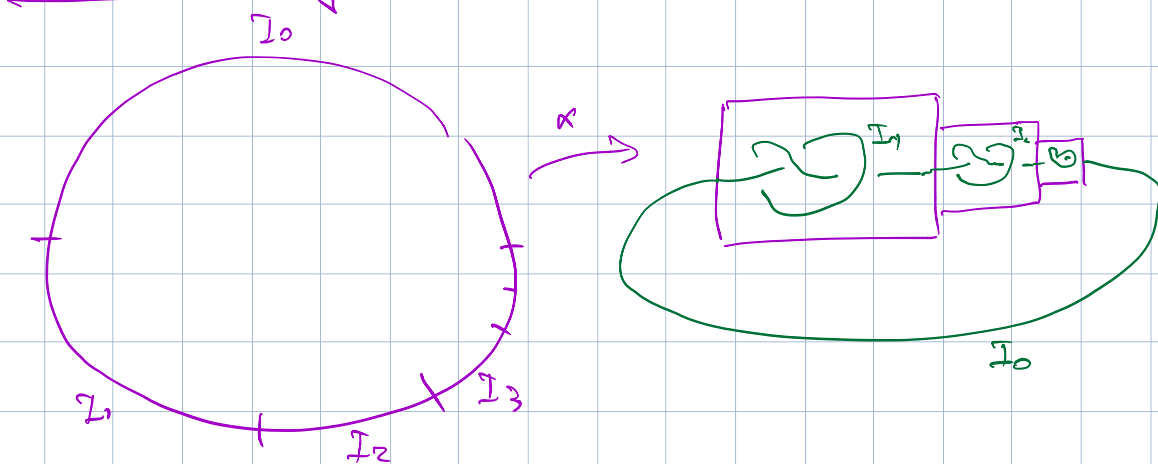
... $S^1 \sqcup \dots \sqcup S^1$ ...

Knot

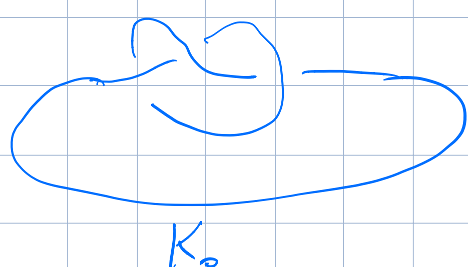
link

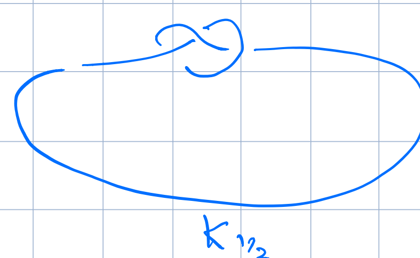
Equivalent $K_0 \sim K_1$ if $\exists (K_t)_{t\in[0,1]}$

Wild embeddings:



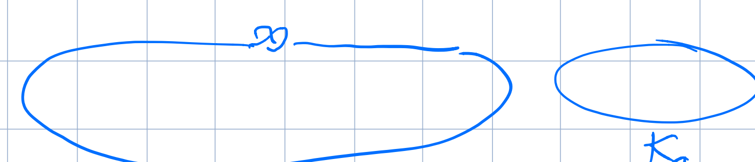
Wild isotopy





$K: S^1\times[0,1] \to \mathbb{R}^3$

$K_{1-\varepsilon}$



Smooth viewpoint

Smooth knot: $\alpha: S^1 \to \mathbb{R}^3$ C^1, $\alpha'(z)\neq 0\ \forall z$

$\widehat{\text{knot}}$ $\alpha: S^1 \to S^3$

$S^3 = \widehat{\mathbb{R}^3} = \mathbb{R}^3 \cup \{\infty\}$

$\alpha_0 \sim_{\widehat{\text{isotopic}}} \alpha_1$ if $\alpha : S^1 \times [0,1] \to \mathbb{R}^3$ $\alpha_t'(z) \neq 0 \ \forall z$
S^3

Remarks:

- every knot is $\widehat{\text{knot}}$
- every $\widehat{\text{knot}}$ is $\widehat{\text{isotopic}}$ to knot
- two knots that are $\widehat{\text{isotopic}}$ are isotopic

$\Longrightarrow$ same theory in $\mathbb{R}^3$ or S^3.

Proposition: if $K \subset S^3$ is image of a knot then it is the image of ≤ 2 / isotopy.

Rem: happens only 1

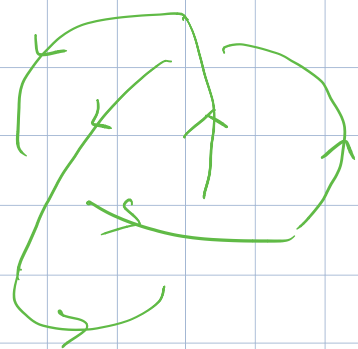
Proof: suppose $\alpha_0, \alpha_1 : S^1 = [0,1]/_{0=1} \longrightarrow K$

- up to rotating one S^1 assume $\alpha_0(0) = \alpha_1(0)$;
- $\alpha_1^{-1} \circ \alpha_0$ self-diffeo of S^1; can assume up to mirroring (not isotopy) one S^1 orientation-preserving i.e. $[0,1] \longrightarrow [0,1]$ increasing with fixed ends.

$\gamma_t(s) = (1-t)s + t \cdot (\alpha_1^{-1} \circ \alpha_0)(s)$

$\alpha_t = \alpha_1 \circ \gamma_t$ isotopy $\alpha_0 \sim \alpha_1$. ▨

So: parametrized knot is oriented



Alternative viewpoints on equivalence:

The following are equivalent for $\alpha_0, \alpha_1 : S^1 \to S^3$

(i) $\exists\, \alpha : S^1 \times [0,1] \to S^3$ s.t. $\alpha_0 = \alpha(\cdot, 0)$ $\alpha_1 = \alpha(\cdot, 1)$

(ii) $\exists\, f : S^3 \times [0,1] \to S^3$ f_t self-diffeo of S^3

$f_0 = id$ $f_1 \circ \alpha_0 = \alpha_1$

(iii) $\exists\, f : S^3 \to S^3$ orientation-pres self-diffeo

s.t. $\alpha_1 = f \circ \alpha_0$

(ii) $\Rightarrow$ (i) $\alpha_t = f_t \circ \alpha_0$

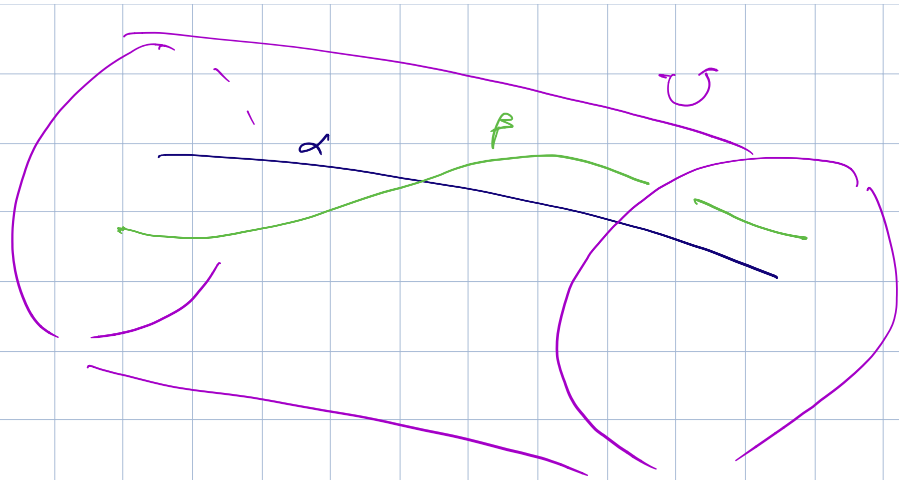
(ii) $\Rightarrow$ (iii) $f = f_1$

(i) $\Rightarrow$ (ii) – ambient isotopy

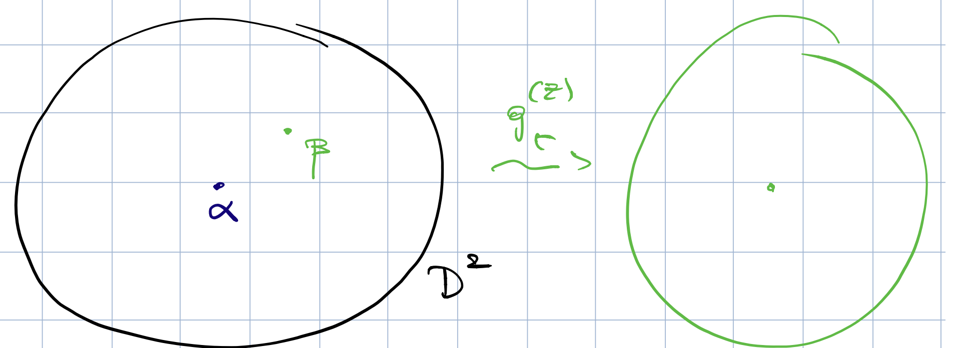
$\exists\, \alpha : S^1 \times [0,1] \to S^3$

Since $[0,1]$ is cpt enough to show that given α any β sufficiently close as C^1 map is ambient isotopic:

take $U \cong S^1 \times D^2$ regular neighborhood with $\alpha(S^1) = S^1 \times \{0\}$



In every $z \times D^2$ I see :

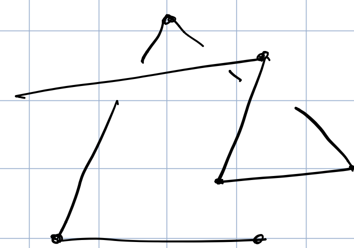


(iii) $\Rightarrow$ (ii) Enough to show that a self-diffeo of S^3 orientation preserving f is isotopic to id.

- wlog assume fixes 0 & ∞
- isotope it on balls of growing radius to 0 $d_0 f \in GL_+(n,\mathbb{R})$
- $GL_+(n,\mathbb{R})$ connected.

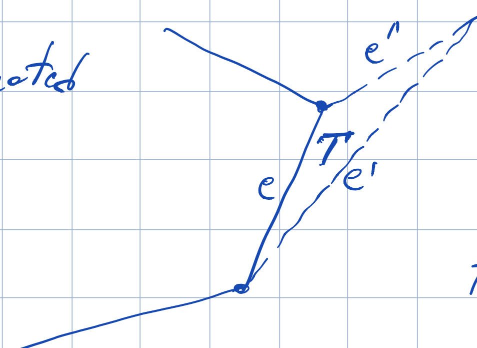
– Piecewise linear viewpoint

knot in $\mathbb{R}^3$: polygonal curve



in S^3 : curve in ~~1~~ 1-skeleton of triangulation of S^3 (S^3 : standard 3-simplex in $\mathbb{R}^4$ allow any subdivision)

Isotopy : generated



$T \cap K = 1$ edge e

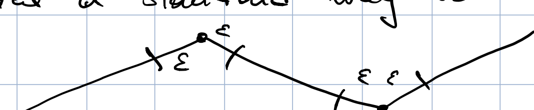
$K \leftrightarrow K \setminus e \cup (e' \cup e'')$

in $\mathbb{R}^3$

in S^3 : same allowing also triangulation of S^3 to change

Fact : these viewpoints are equivalent. Strategy :

- define a standard way to smoothen



including a parametrization

$\underset{PL}{K} \longrightarrow s_\varepsilon(K)$ for $0 < \varepsilon << 1$

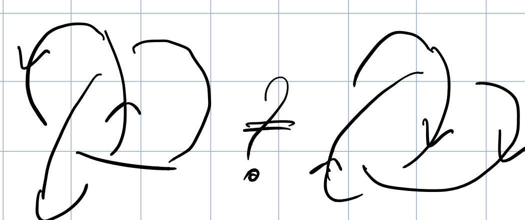
- show that all $s_\varepsilon(K)$ are smoothly isotopic
- show that $K_0 \sim_{PL} K_1$ then $s_\varepsilon(K_0) \sim_{\text{smooth}} s_\varepsilon(K_1)$
- given α smooth and K a very good PL approx of α show that $\alpha \sim_{\text{smooth}} s_\varepsilon(K)$
- α_0, α_1 smoothly isotopic $\Rightarrow$ K_0, K_1 very good approx are PL isotopic

———— o ————

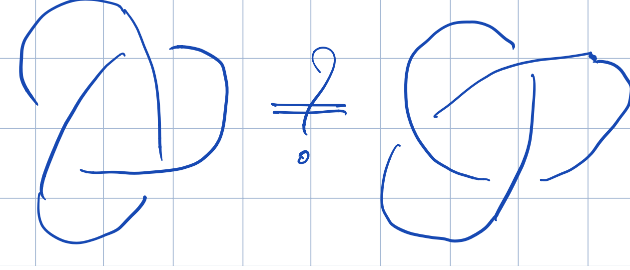
2 viewpoints : oriented or not

2 viewpoints : can decide to define $K \sim f(K)$ even with $f: S^3 \to S^3$ orient-rev.

Knot : invertible?



chiral?



yes : amphichiral

no : chiral (left + right version)

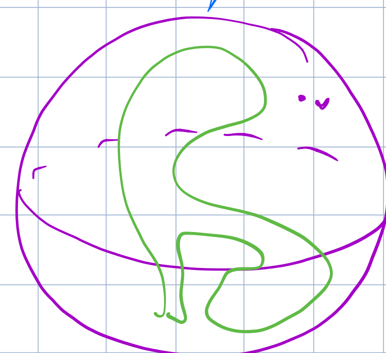
Knots + isotopy are represented by projections (planar diagrams) + moves.

<u>Smooth viewpoint</u> : $\alpha : S^1 \to \mathbb{R}^3$

v vector in $\mathbb{R}^3$ $\quad \pi_v \quad \perp$ projection on $v^\perp$

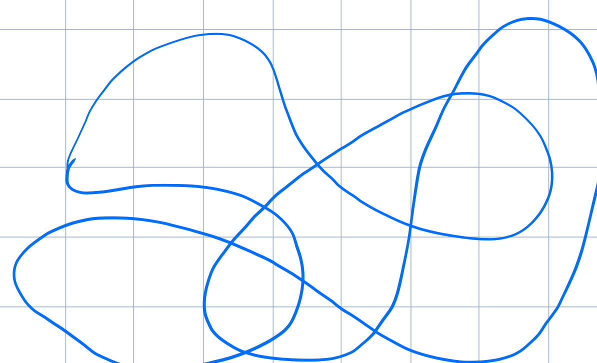
up to small perturbation of v will have:

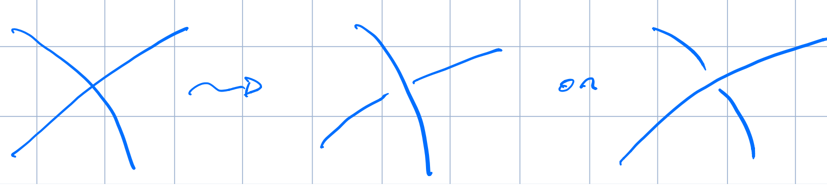
- $(\pi_v \circ \alpha)'(z) \neq 0 \ \forall z$ ($\Rightarrow$ immersion of S^1 in $\mathbb{R}^2$)
- transverse double pts only



$\dfrac{\alpha'}{\|\alpha'\|}$

$\Rightarrow (\pi_v \circ \alpha)(S^1)$

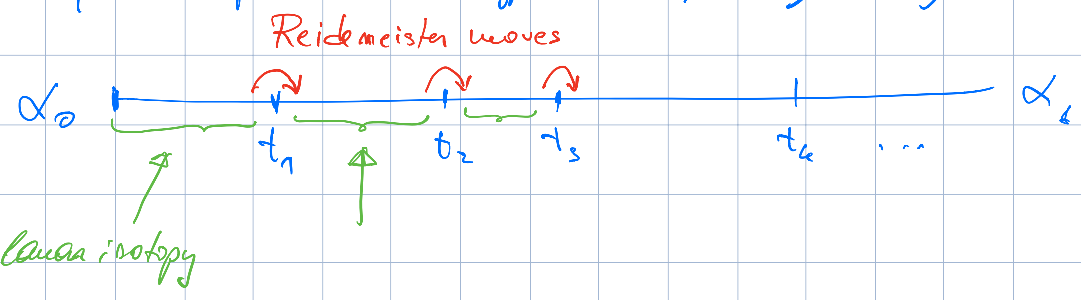


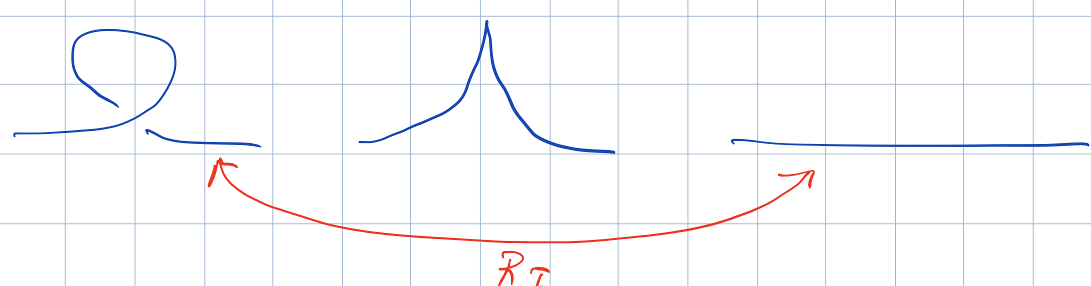


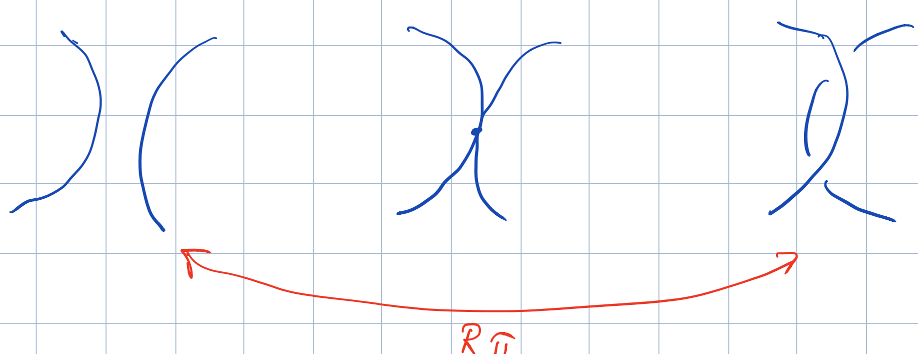
Fact: every knot has a diagram; every diagram gives a knot / isotopy.

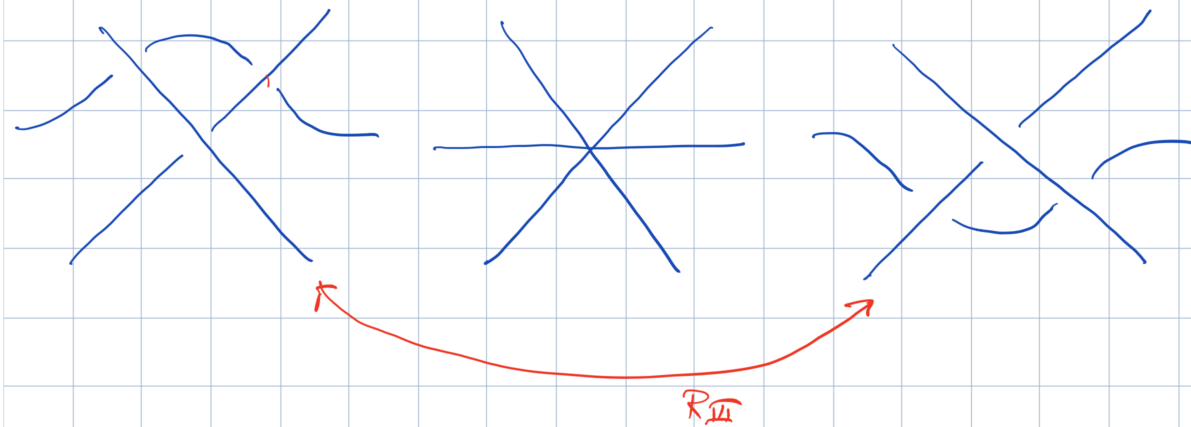
Fact: given $(\alpha_t)_{t \in [0,1]}$ an isotopy of knots up to small perturbation of v I have that

- $(\pi_v \circ \alpha_t)$ has at most
  - 1 point with 0 derivative
  - 1 non-transverse double pt
  - 1 transverse triple point
- previous phenomena happen at finitely many t's.







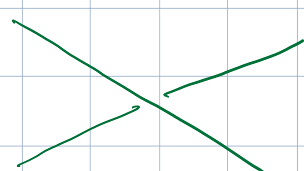


Theorem: two diagrams represent isotopic knots $\iff$ related by planar isotopy plus R_I, R_{II}, R_{III}.

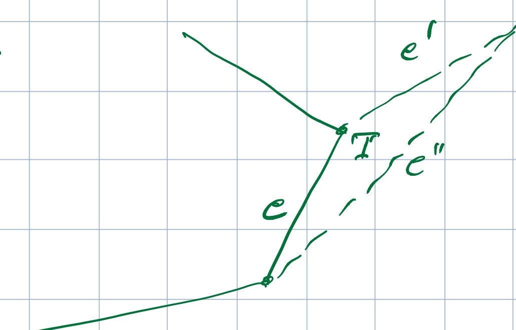
Facts:
- true also for links
- oriented version valid

PL viewpoint: $K \subset \mathbb{R}^3$

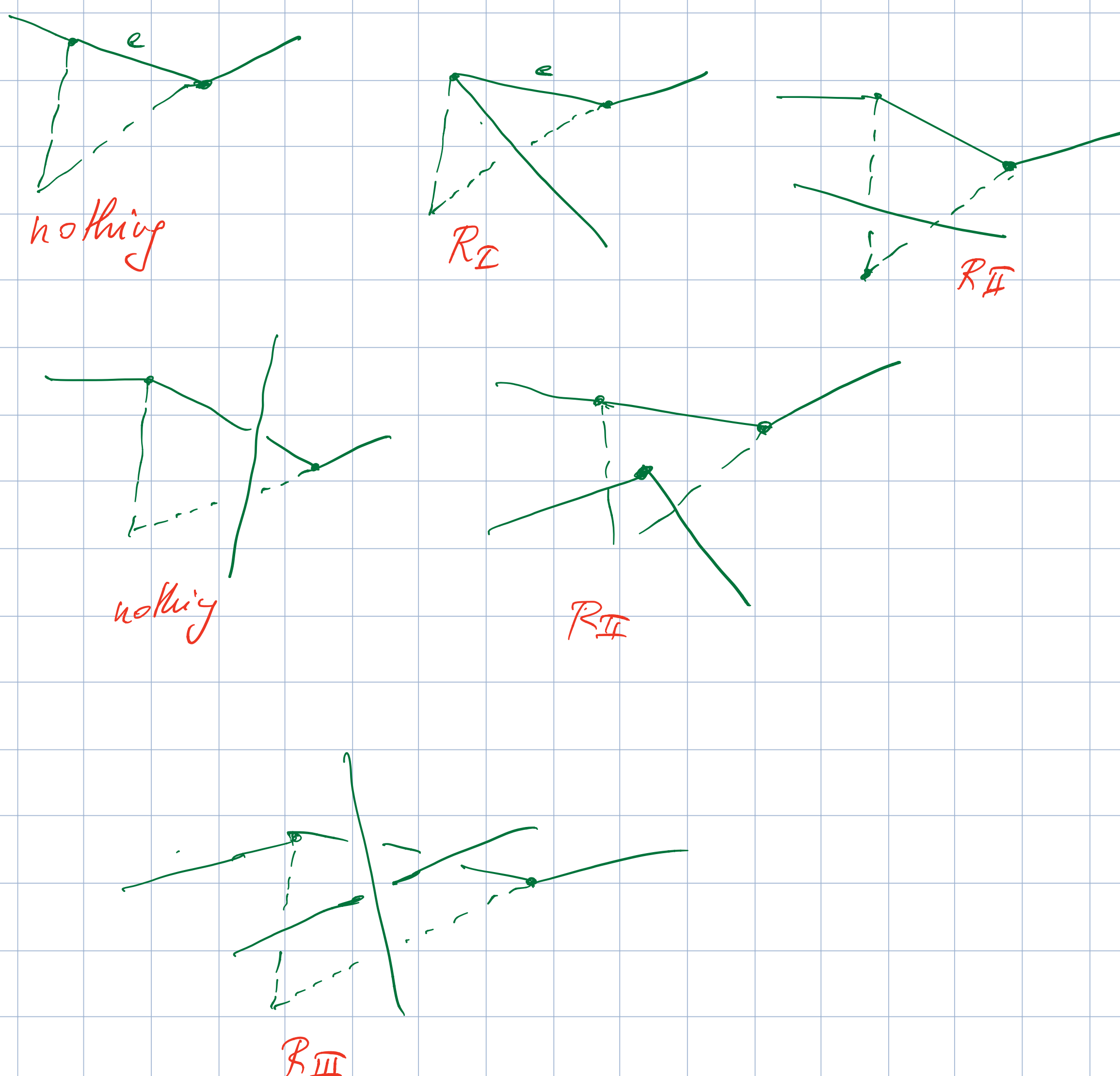
- wlog π_v send every segment of K to segment, vertices to different points non on segments and only transverse double pts



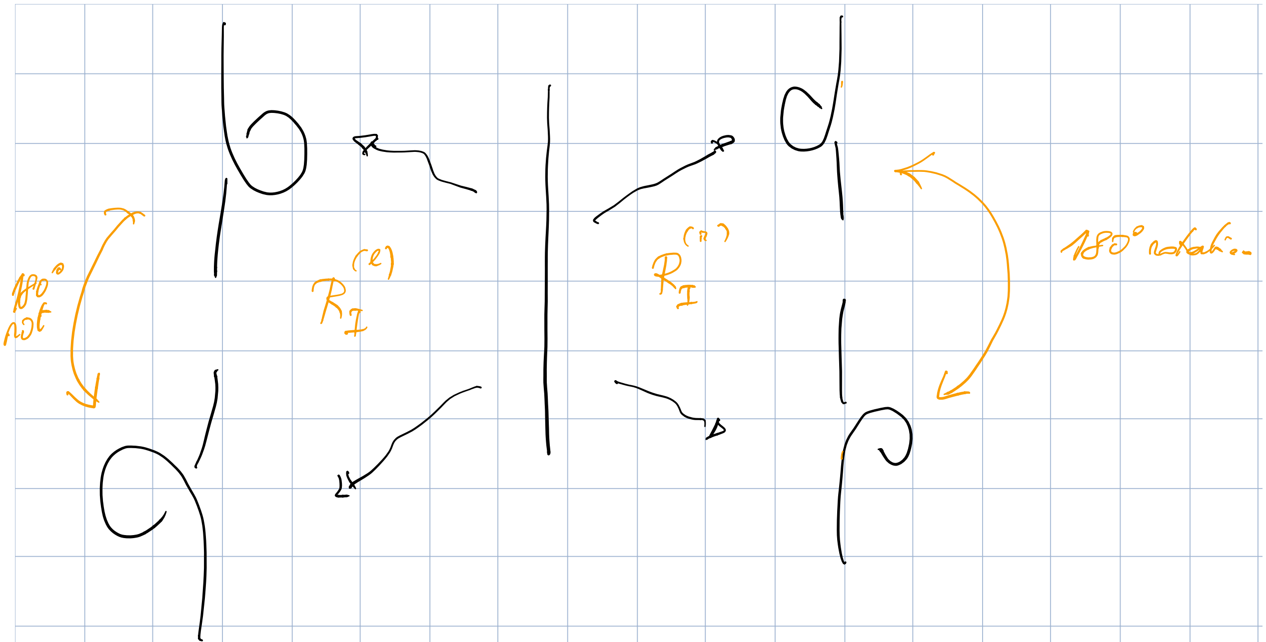
- if have



- can assume T projects to triangle
- up to subdividing assume $\pi_v(T)$ contains ≤ 1 vertex or ≤ 1 crossing:

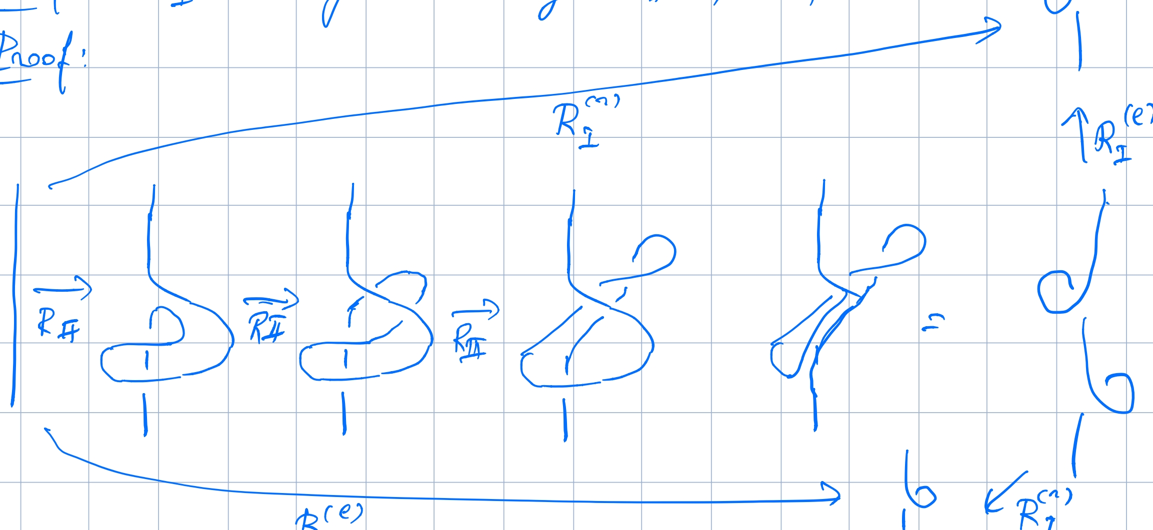


Thm: every PL planar diagram represents a knot/isotopy.
Isotopy $\Longleftrightarrow$ planar isotopy $+ R_I, R_{II}, R_{III}$.



Prop: $R_I^{(l)}$ is generated by $R_I^{(r)}$, R_{II}, R_{III}

Proof:



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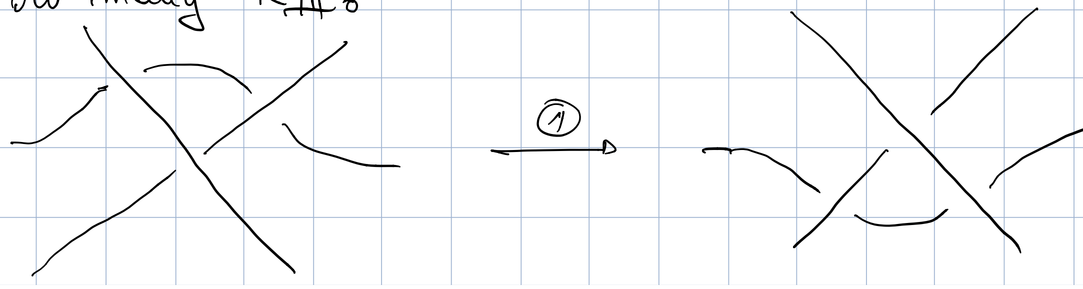
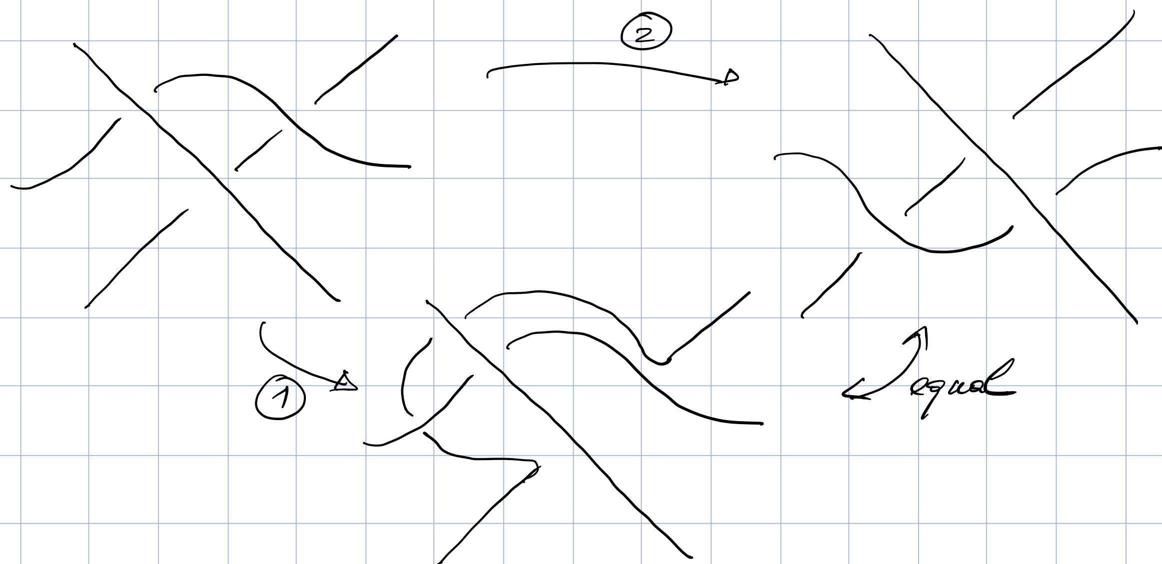
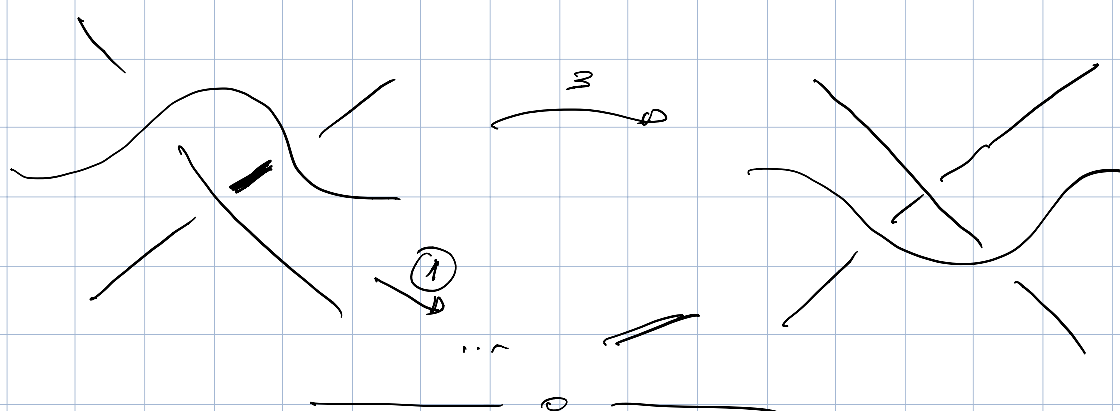
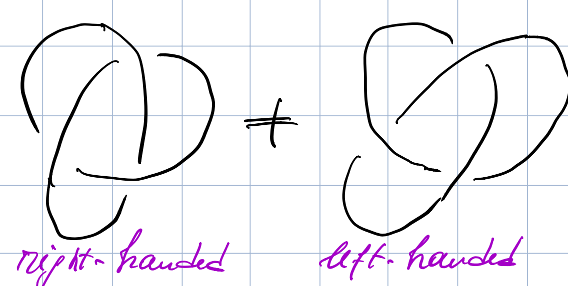
How many R_{II}?





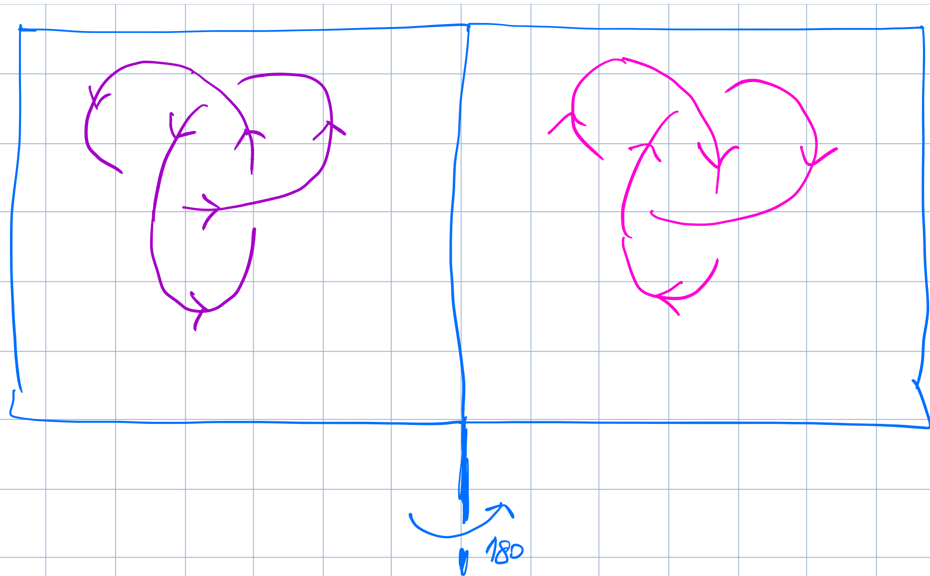


<u>Fact</u>: trefoil is chiral



Prop: trefoil is invertible

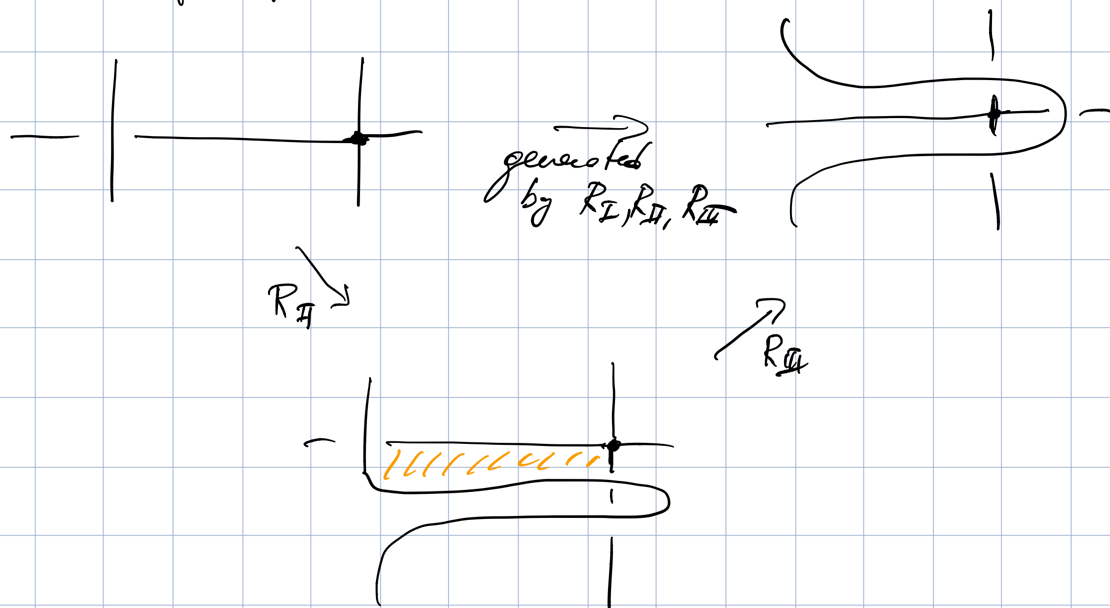
<u>Proof</u>:


180

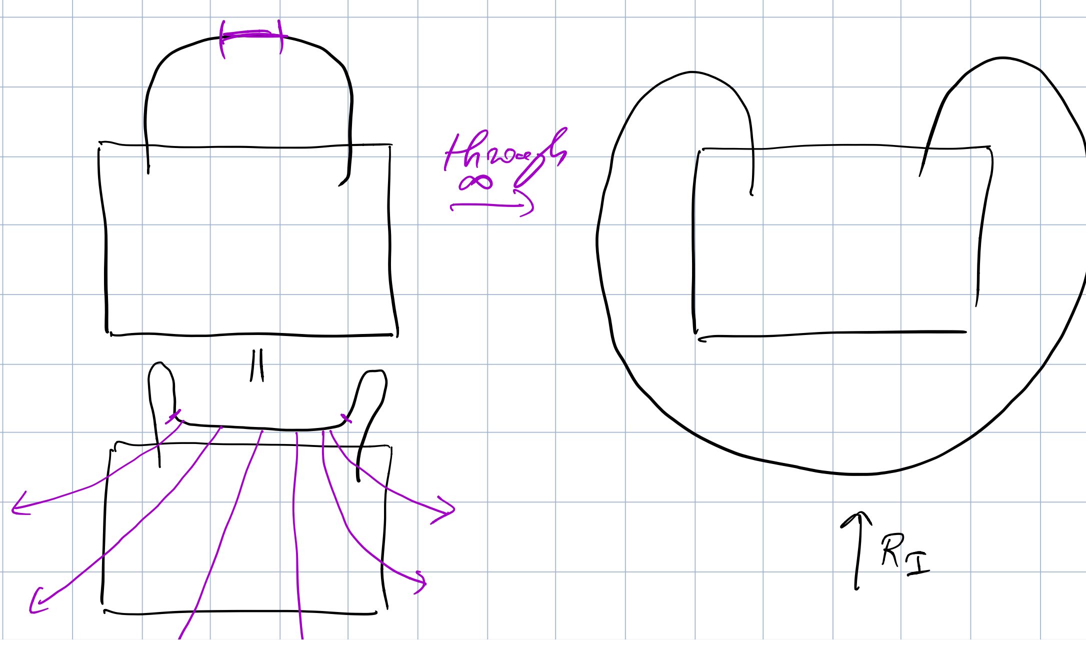
Exercise: express this isotopy using R_I, R_{II}, R_{III}

Two useful facts:

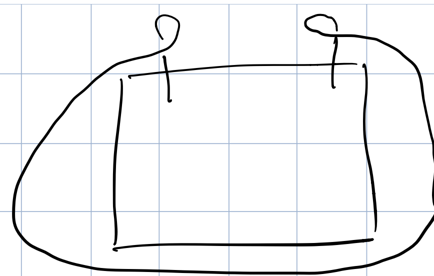
1)



2) isotopy of diagram on S^2 (rather than $\mathbb{R}^2$) is generated by R_I, R_{II}, R_{III}:

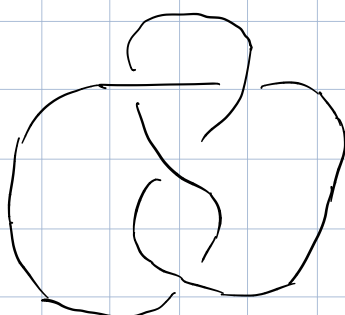


R_{II}, R_{III}

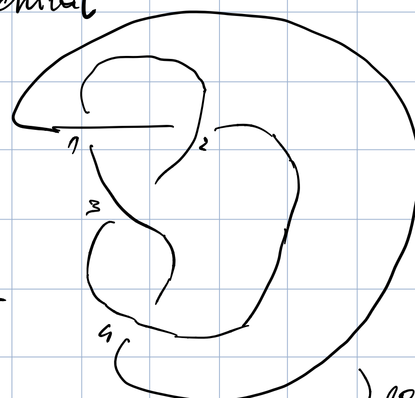


Prop: the figure-8 knot is amphichiral

Proof 1:

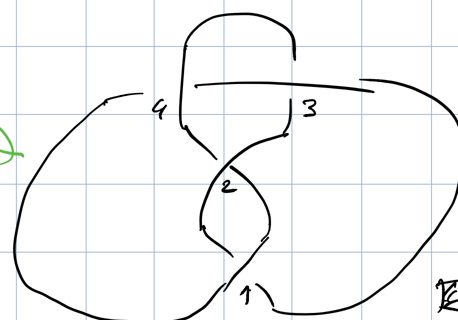


isotopy on S^2



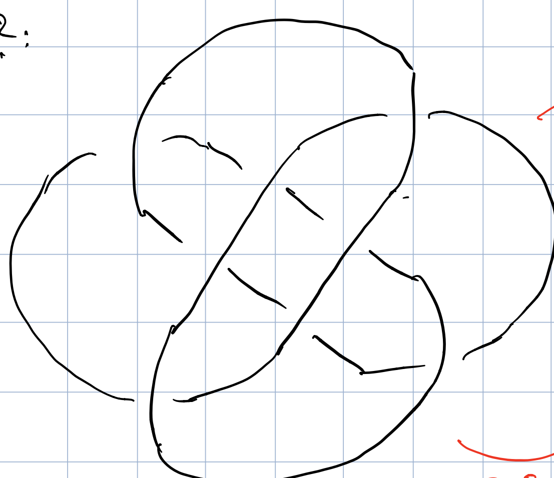
rotation $180°$

mirror



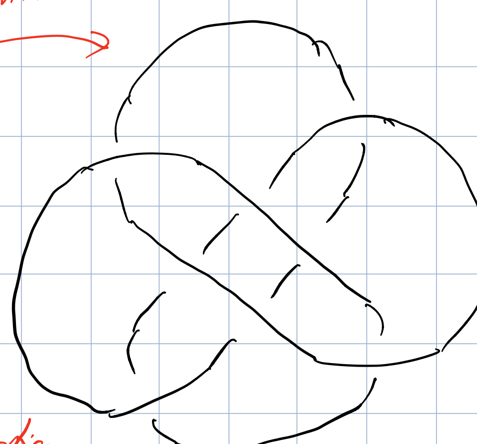
□

Proof 2:



mirror

90° rotation



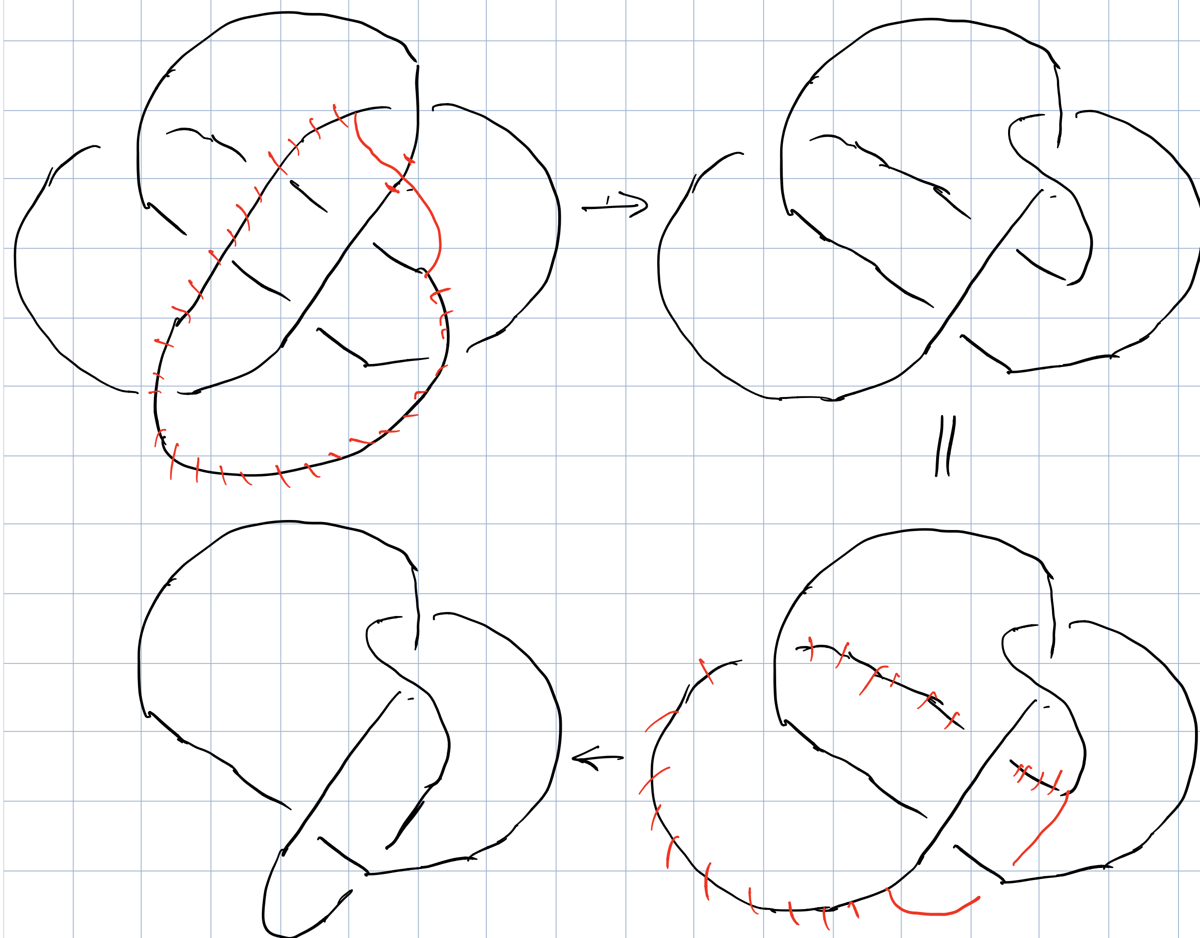


figure-8.

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