The state sum formula for V(L).

Titolo nota 19/05/2019

In the lesson given on 14th May,
I provided an argument for the equality

$$V(L) = \left(-t^{\frac{1}{4}}\right)^{3w(D)} \langle D \rangle ,$$

where $\langle D \rangle = \sum_{s} \langle D | s \rangle$ is the Kauffman bracket, defined by the formula

$$\langle D | s \rangle = \left(t^{-\frac{1}{4}}\right)^{\sum_i s(i)} \cdot \left(-t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)^{|sD|-1}$$

where the sum varies over all the states of D, and D is any diagram of L.

The argument was based on the following fact, which easily follows from the fact that every diagram may be turned into a diagram of the unlink by changing the overpass/underpass information at some of its crossings.

Proposition: let $V, W : \left\{ \begin{matrix} \text{oriented} \\ \text{diagrams} \end{matrix} \right\} \to \mathbb{Z}[t, t^{-1}]$

be such that:

① $V(O) = W(O) = 1$

② $t^{-1} V(\ ) - t V(\ ) + (t^{-\frac{1}{2}} - t^{\frac{1}{2}}) V(\ ) = 0$

$$t^{-1} W(\nearrow\!\!\!\!\nwarrow) - t\, W(\nwarrow\!\!\!\!\nearrow) + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})\, W(\uparrow\,\uparrow) = 0$$

③ Both V and W are link invariants, i.e. they are invariant w.r.t. Reidemeister moves.

Then, $V = W$.

I got (erroneously) convinced that, if V, W satisfy all the above requirements but – possibly – the fact that W is a link invariant, then the conclusion would follow anyway (because W would be forced to be a link invariant). However, I see no reason why this should be true.
Therefore, in order to fill the gap, we are left to show that the quantity

$$W(D) = \left(-t^{\frac{1}{4}}\right)^{3w(D)} \langle D \rangle$$

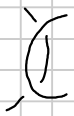
is indeed invariant w.r.t. the Reidemeister moves.

Remark: In the literature after Kauffman, one usually defines the Jones polynomial by the formula defining $W(L)$, after checking that such formula is indeed invariant w.r.t. the Reidemeister moves. Therefore, what we are going to do is just proving that the above

formula can be used to define a polynomial invariant of links, the Jones polynomial.
It will be clear that this construction of the Jones polynomial is much easier than the proof that the HOMFLY polynomial is well-defined.

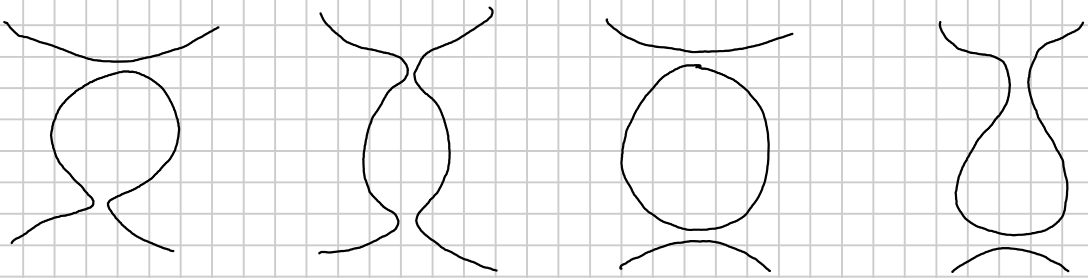
We begin with the easiest control, i.e. the fact that $W(D)$ is invariant w.r.t. Reidemeister moves of type II. Let D, D' be diagrams which are obtained one from the other via a Reidemeister move of type II, let $\{2,3,\dots,n\}$ be the crossings of D, and $\{0,1,\dots,n\}$ the crossings of D'. We treat the case described in following picture:

D $\qquad$ D'

the other one — i.e. when the local picture of D' is  — being identical.
If s is a state of D, we denote by $s_{++}, s_{+-}, s_{-+}, s_{--}$ the states of D' assigning the same sign as s to every crossing $i \geq 2$, and s.t. $s_{++}(0) = s_{++}(1) = 1$, $s_{+-}(0) = 1$, $s_{+-}(1) = -1$,

$s_{-+}(0) = -1$, $s_{-+}(1) = 1$, $s_{--}(0) = s_{--}(1) = -1$.

We have:



$s_{++}D'$ $\quad$ $s_{+-}D'$ $\quad$ $s_{-+}D'$ $\quad$ $s_{--}D'$

Note that $s_{+-}D' = D$, while $s_{++}D' = s_{--}D'$ and $s_{-+}D'$ is obtained from $s_{++}D' = s_{--}D'$ by adding a (trivial) component. Also observe that

$\sum_{i\geq 0} s_{++}(i) = 2 + \sum_{i\geq 2} s(i)$, $\sum_{i\geq 0} s_{--}(i) = -2 + \sum_{i\geq 2} s(i)$

$\sum_{i\geq 0} s_{+-}(i) = \sum_{i\geq 0} s_{-+}(i) = \sum_{i\geq 2} s(i)$. Thus

$\langle D' \mid s'_{+-} \rangle = \langle D \mid s \rangle$, while

$$\langle D' \mid s_{++} \rangle + \langle D' \mid s_{--} \rangle + \langle D' \mid s_{-+} \rangle =$$

$$= \left(t^{-\frac{1}{4}}\right)^{\sum_{i\geq 0} s_{++}(i)} \left(-t^{-\frac{1}{2}} - t^{\frac{1}{2}}\right)^{|s_{++}D'|-1} +$$

$$+ \left(t^{-\frac{1}{4}}\right)^{\sum_{i\geq 0} s_{--}(i)} \left(-t^{-\frac{1}{2}} - t^{\frac{1}{2}}\right)^{|s_{--}D'|-1} +$$

$$+ \left(t^{-\frac{1}{4}}\right)^{\sum_{i\geq 0} s_{-+}(i)} \left(-t^{-\frac{1}{2}} - t^{\frac{1}{2}}\right)^{|s_{-+}D'|-1} =$$

$$= \left[\left(-t^{-\frac{1}{2}} - t^{\frac{1}{2}}\right)^{|s_{++}(D')|-1}\right] \left(t^{-\frac{1}{4}}\right)^{\sum_{i\geq 2} s(i)} \cdot$$

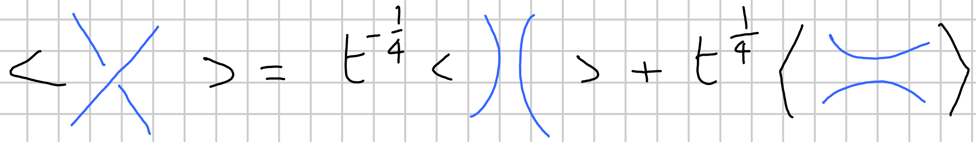
$$\cdot \left(\left(t^{-\frac{1}{4}}\right)^{2} + \left(t^{-\frac{1}{4}}\right)^{-2} + \left(-t^{-\frac{1}{2}} - t^{\frac{1}{2}}\right) \right) = 0$$

Thus,

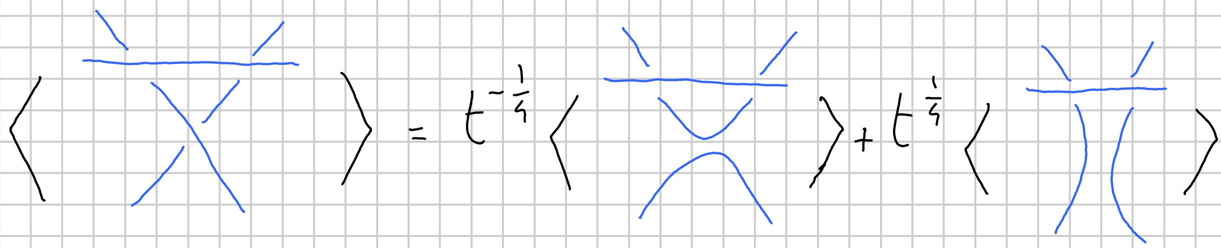
$$\langle D \mid s \rangle = \langle D' \mid s_{++} \rangle + \langle D' \mid s_{+-} \rangle + \langle D' \mid s_{-+} \rangle + \langle D' \mid s_{--} \rangle$$

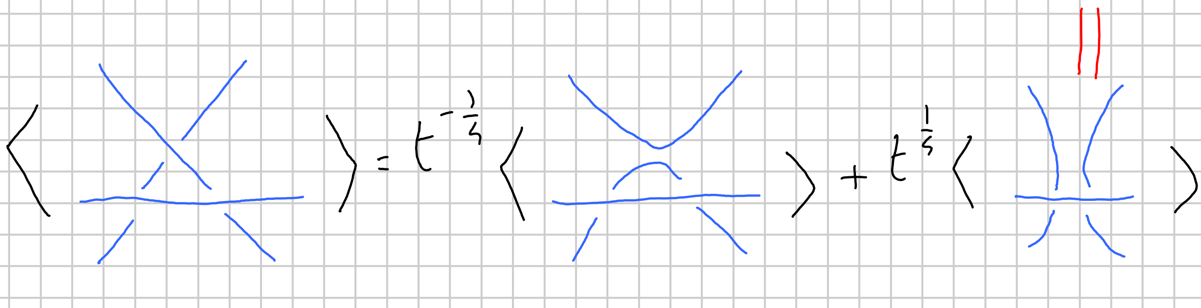
By summing over all the statuses of D, we get $\langle D \rangle = \langle D' \rangle$. Since $w(D) = w(D')$, this finally gives $W(D) = W(D')$, as desired.

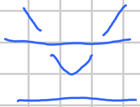
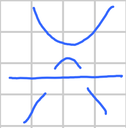
Let us now examine a Reidemeister move of type III. First observe that the Kauffman bracket satisfies the skein relation



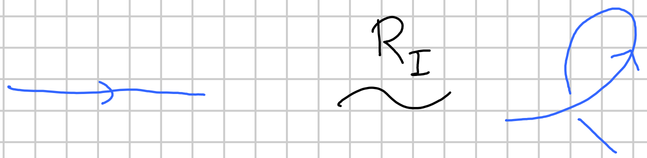
In fact, if D is the diagram on the left-hand side, then the first summand on the right-hand side represents the contribution given to $\langle D \rangle$ by all the statuses for which $s(i) = 1$, while the second summand represents the contribution given by the statuses for which $s(i) = -1$ (here i is the label of the crossing we are looking at). Therefore,





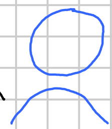
The diagrams  and  are equivalent via Reidemeister moves of type II. Therefore, as proved above, they have the same Kauffman bracket. This implies that the left-hand sides of the two equalities above are equal. Thus, if D, D' are the diagrams described on the right-hand sides, then $\langle D \rangle = \langle D' \rangle$. Since $w(D) = w(D')$, we finally have $W(D) = W(D')$, as desired.

We are only left to check the invariance of W w.r.t. Reidemeister moves of type I. We only consider the move



D $\qquad\qquad$ D'

First observe that D' has one more negative crossing than D, hence $w(D') = w(D) - 1$.
Moreover,

$\langle D' \rangle = \langle$  $\rangle = t^{-\frac{1}{4}} \langle$  $\rangle + t^{\frac{1}{4}} \langle$  $\rangle =$

$= t^{-\frac{1}{4}} \langle D \rangle + t^{\frac{1}{4}} \langle D \cup \bigcirc \rangle .$

Observe that, if we add to D a trivial component to get $D \cup \bigcirc$, then the states of D canonically correspond to the states of $D \cup \bigcirc$. Moreover, for any such states s, $s(D \cup \bigcirc) = s(D) \cup \bigcirc$, hence

$|s(D \cup \bigcirc)| = |s(D)| + 1$, and

$$\langle D \cup \bigcirc \rangle = \left(-t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) \langle D \rangle .$$

Coming back to the relation above, we thus get

$\langle D' \rangle = \langle$  $\rangle = t^{-\frac{1}{4}} \langle D \rangle + t^{\frac{1}{4}} \left(-t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) \langle D \rangle =$

$$= -t^{\frac{3}{4}} \langle D \rangle ,$$

hence

$$W(D') = \left(-t^{\frac{1}{4}}\right)^{3w(D')} \langle D' \rangle = \left(-t^{\frac{1}{4}}\right)^{3w(D)} \left(-t^{\frac{1}{4}}\right)^{-3} \left(-t^{\frac{3}{4}}\right) \langle D \rangle$$

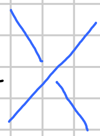
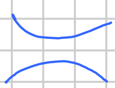
$$= \left(-t^{\frac{1}{4}}\right)^{w(D)} \langle D \rangle = W(D).$$

The proof that $W(D)$ is invariant w.r.t. Reidemeister moves is now complete.

Remark: The Kauffman bracket is often DEFINED as the unique polynomial invariant of (unoriented) diagrams such that:

① $\langle \bigcirc \rangle = 1$

② $\langle D \cup \bigcirc \rangle = \left(-t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) \langle D \rangle$

③ $\langle$  $\rangle = t^{-\frac{1}{4}} \langle$  $\rangle + t^{\frac{1}{4}} \langle$  $\rangle$

Here we proved that indeed $\langle \cdot \rangle$ satisfies ② and ③ (① is obvious).