IL TEOREMA DI ESISTENZA DI TONELLI

TEO: SIA L UNA LAGRANGIANA T.C.

① $L(x,y,z)$ e $L_z(x,y,z)$ CONTINUE

② $z \to L(x,y,z)$ CONVESSA $\forall x,y$

③ $L(z) \geq \psi(z)$ DOVE $\psi : \mathbb{R} \to \mathbb{R}$ È T.C. $\lim_{z \to \pm\infty} \frac{\psi(z)}{|z|} = +\infty$.

$\Rightarrow$ $\mathcal{L}(u) = \int_a^b L(x, u(x), u'(x))\, dx$

AMMETTE MINIMO IN $A = \{ u \in W^{1,1}(a,b) : u(a) = \alpha,\ u(b) = \beta \}$ $\forall \alpha, \beta \in \mathbb{R}$.

[NO DIM.]

ESEMPI:

$$\mathcal{L}(u) = \int_a^b |u'|^2 + (u-f)^2 \, dx \qquad f \in L^2(a,b), \quad u \in W^{1,2}(a,b)$$

$\mathcal{L}$ STRETT. CONVESSA IN (u, u')

$\Rightarrow \forall \alpha, \beta \ \exists!$ MINIMO $u \in W^{1,2}(a,b)$ T.C. $u(a) = \alpha$, $u(b) = \beta$.

Inoltre è VERIFICATA L'EQUAZIONE DI E.L. IN SENSO DEBOLE

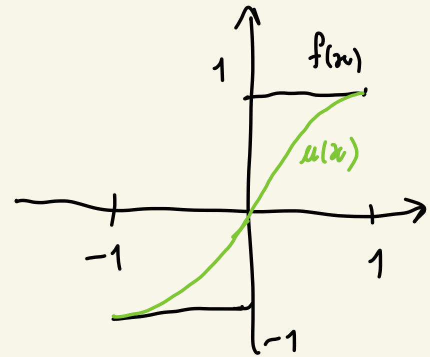
$$\boxed{\begin{cases} u' = g \in W^{1,1} \\ g' = u - f \end{cases}} \Rightarrow g \in C^0 \Rightarrow u \in C^1$$

SE f È CONTINUA $\Rightarrow g \in C^1 \Rightarrow u \in C^2$ E SI HA

$$\boxed{-u'' + u - f = 0} \qquad \boxed{u(a) = \alpha, \quad u(b) = \beta}$$

ESEMPIO: $a = -1, \quad b = 1 \qquad f = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} \qquad f(x) = \dfrac{x}{|x|}$

$$\mathcal{L}(u) = \int_{-1}^{1} |u'|^2 + \left(u - \frac{x}{|x|}\right)^2 dx$$



DATI AL BORDO: $u(-1) = -1, \quad u(1) = 1.$

$$\begin{cases} u \in C^1(-1,1) \\ u'' - u + 1 = 0 \qquad x > 0 \\ u' - u - 1 = 0 \qquad x < 0 \\ u(-1) = -1 \\ u(1) = 1 \end{cases}$$

OMOGENEA ASSOCIATA $\quad u'' - u = 0 \qquad u = c_1 e^{x} + c_2 e^{-x}$

1. $x \geq 0$ $\qquad u_p = 1$ $\qquad u^+(x) = c_1 e^{x} + c_2 e^{-x} + 1$

$u^+(1) = 1 \qquad c_1 e + \frac{c_2}{e} = 0 \Rightarrow c_2 = -c_1 e^2$

$$\boxed{u^+(x) = c_1 e^{x} - c_1 e^2 e^{-x} + 1} \qquad c_1 \in \mathbb{R}$$

2. $x < 0$ $\qquad u_p = -1$ $\qquad u^- = d_1 e^{x} + d_2 e^{-x} - 1$

$u^-(-1) = -1 \qquad \frac{d_1}{e} + d_2 e = 0 \qquad d_2 = -\frac{d_1}{e^2}$

$$\boxed{u^-(x) = d_1 e^{x} - d_1 \frac{e^{-x}}{e^2} - 1}$$

VOGLIAMO CHE $u^+(0) = u^-(0)$ E $u^{+\prime}(0) = u^{-\prime}(0)$.

$u^+(0) = c_1 - c_1 e^2 + 1 = c_1(1 - e^2) + 1 = u^-(0) = d_1\left(1 - \frac{1}{e^2}\right) - 1$

$c_1(e^2 - 1) + d_1 \frac{e^2 - 1}{e^2} = 2 \qquad \boxed{c_1 e^2 + d_1 = 2\frac{e^2}{e^2 - 1}}$

$$u^{+\prime}(0) = c_1 + c_1 e^2 = c_1(1+e^2) = u^{-\prime}(0) = d_1 + \frac{d_1}{e^2} = d_1\left(\frac{1+e^2}{e^2}\right)$$

$$\boxed{c_1 e^2 = d_1}$$

$$\Rightarrow \begin{cases} c_1 = \dfrac{1}{e^2-1} \\ d_1 = \dfrac{e^2}{e^2-1} \end{cases} \Rightarrow \quad u(x) = \begin{cases} \dfrac{e^x}{e^2-1} - \dfrac{e^2 e^{-x}}{e^2-1} + 1 & x \geq 0 \\ \dfrac{e^2 e^x}{e^2-1} - \dfrac{e^{-x}}{e^2-1} - 1 & x \leq 0 \end{cases}$$

ESEMPIO (MANIÀ):

$$\boxed{\mathcal{L}(u) = \int_0^1 (u^3 - x)^2 |u'|^6 \, dx}$$ CON COND. $\boxed{u(0)=0, \ u(1)=1}$

$$L(x,y,z) = (y^3 - x)^2 |z|^6 \geq 0 \quad \forall (x,y,z)$$

L REGOLARE (C^1)

$L \not\geq a|u'|^\sigma + b$ PER NESSUN $\sigma > 0$.

$\mathcal{L}(u) = 0 \iff u = \sqrt[3]{x}$ MINIMO ASSOLUTO (UNICO, NON C^1)

$u(x) = \sqrt[3]{x} \notin W^{1,6}$ $\quad u'(x) = \frac{1}{3} x^{-\frac{2}{3}} \Rightarrow \int_0^1 |u'|^\sigma = \frac{1}{3^\sigma} \int_0^1 \frac{1}{x^{\frac{2\sigma}{3}}} dx < +\infty$

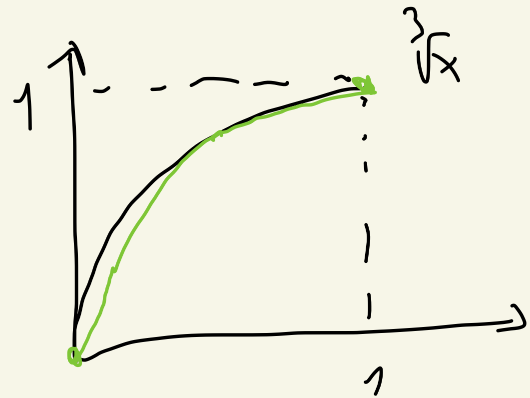
$\iff \sigma < \frac{3}{2}$

$$\boxed{u \in W^{1,\sigma} \iff \sigma \in \left(0, \tfrac{3}{2}\right)}$$

PROP: $\exists\, c > 0$ T.C. $\mathcal{L}(u) > c \quad \forall\, u \in Lip([0,1])$ T.C. $u(0) = 0$, $u(1) = 1$.

IN PART. $\boxed{\inf_{LIP} \mathcal{L} \geq c > 0 = \inf_{W^{1,1}} \mathcal{L}}$

FENOMENO DI LAURENTIEV



[NO DIM.]

OSS: CI SONO ESEMPI DI $\mathcal{L}$ T.C. $\inf_{LIP} \mathcal{L} < \inf_{C^1} \mathcal{L}$.