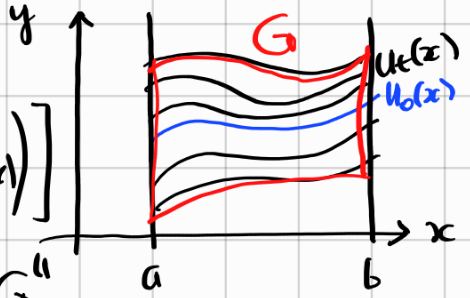
ELEMENTI di CALCOLO delle VARIAZIONI
LEZIONE 7 - 18.3.2024

Calibrazioni

Prendo una famiglia di soluzioni dell'equazione di EL: $u_t(x)$:



$$\frac{\partial L}{\partial y}\left(x, u_t(x), u_t'(x)\right) = \frac{d}{dx}\left[\frac{\partial L}{\partial z}\left(x, u_t(x), u_t'(x)\right)\right]$$

$u_t(x)$ "fibrano la regione G", cioè $f(x,t) = (x, u_t(x))$, $t \in I$

$f : [a,b] \times I \longrightarrow G$ sia un diffeomorfismo

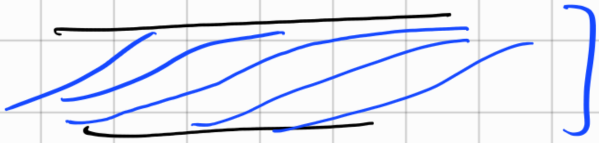
$L = L(x,y,z)$ $\qquad \forall (x,y) \quad z \longmapsto L(x,y,z)$ convessa.

Allora posto $\mathcal{L}(u) = \int_a^b L(x, u(x), u'(x))$

$\forall t \in I \quad u_t$ è minimo per $\mathcal{L}(u)$ tra tutte le $u \in C^1$ con $(x,u(x)) \in G$

con $\boxed{u(a) = u_t(a), \quad u(b) = u_t(b).}$

[Es. transizione di fase, vista la volta scorsa:



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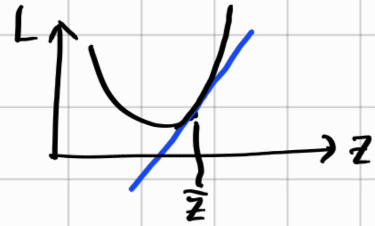
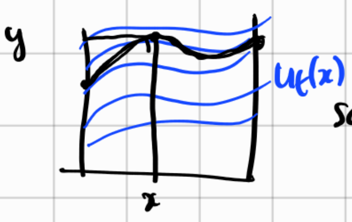
COME SI DIMOSTRA



Passo 1 $\quad L$ convessa in z:

$$L(x,y,z) \geq \overbrace{L(x,y,\bar{z}(x,y)) + \frac{\partial L}{\partial z}(x,y,\bar{z}(x,y)) \cdot (z - \bar{z})}^{M(x,y,z)}$$



scelgo: $\bar{z}(x,y) = u_t'(x)$ con t tale che $u_t(x) = y$

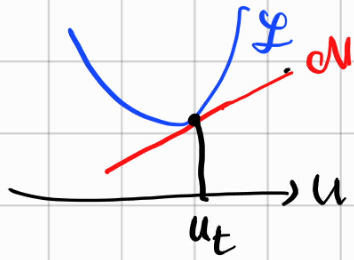
quindi $\bar{z}(x, u_t(x)) = u_t'(x)$.

$$M(x,y,z) = L(x,y,\bar z(x,y)) + \frac{\partial L}{\partial z}(x,y,\bar z(x,y))\cdot(z-\bar z)$$

$$\mathcal{M}(u) = \int_a^b M(x,u(x),u'(x))\,dx$$

$$\mathcal{L}(u) = \int_a^b L \geq \int_a^b M = \mathcal{M}(u)$$

$$\mathcal{L}(u_t) = \int_a^b L(x,u_t(x),\underbrace{u_t'(x)}_{=\,\bar z(x,u_t(x))})\,dx = \int_a^b M(x,u_t(x),u_t'(x))\,dx = \mathcal{M}(u_t)$$



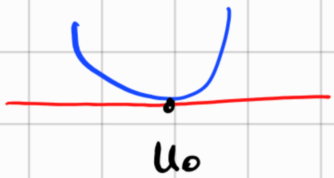
Lemma bordo: Se u_t è minimo per $\mathcal{M}$
allora

$$\mathcal{L}(u) \geq \mathcal{M}(u) \geq \mathcal{M}(u_t) = \mathcal{L}(u_t)$$

u_t è minimo per $\mathcal{L}$.

Passo 2 Verificare che M è una "Lagrangiana nulla".

$$M(x,u(x),u'(x)) \overset{?}{=} \frac{d}{dx}\left[S(x,u(x))\right] \qquad (*)$$



$$\mathcal{M}(u) = \int_a^b M(x,u(x),u'(x))\,dx = \int_a^b \frac{d}{dx}\left(S(x,u(x))\right)dx$$

$$= \left[S(x,u(x))\right]_a^b = S(b,u(b)) - S(a,u(a)).$$

$\mathcal{M}(u)$ dipende solo dai valori $u(a)$ e $u(b)$
quindi ogni funzione è minimo per $\mathcal{M}$ con il suo dato al bordo.
In particolare u_t sarà minimo (con il suo dato al bordo).

Passo 3 Mostrare che se u_t soddisfa E.L. $\forall t \in I$
allora vale $(*)$.

$$M(x,y,z) = \boxed{L(x,y,\bar z(x,y)) + \frac{\partial L}{\partial z}(x,y,\bar z(x,y))\cdot(z-\bar z(x,y))}$$

$$\mathcal{M}(u) = \int_a^b M(x,u(x),u'(x))\,dx = \int_\gamma \omega \quad \text{se pongo:}$$

$$\gamma(t) = (t, u(t)) \qquad \gamma'(t) = \begin{pmatrix} 1 \\ u'(t)\end{pmatrix} \qquad \begin{cases} x = t \\ y = u(t)\end{cases} \quad \begin{cases} dx = dt \\ dy = u'(t)\,dt\end{cases}$$

$$\omega(x,y) = \left[L(x,y,\bar z(x,y)) - \bar z(x,y)\cdot\frac{\partial L}{\partial z}(x,y,\bar z(x,y))\right]dx + \frac{\partial L}{\partial z}(x,y,\bar z(x,y))\,dy$$

$$dx = 1\cdot dt$$

$$dy = u'(t)\,dt \qquad \underline{\text{INFATTI:}}$$

$$\int_\gamma \omega = \int_a^b \Big[L(t,u(t),\bar z(t,u(t))) - \bar z(t,u(t))\cdot\frac{\partial L}{\partial z}(t,u(t),\bar z(t,u(t)))\cdot 1 +$$

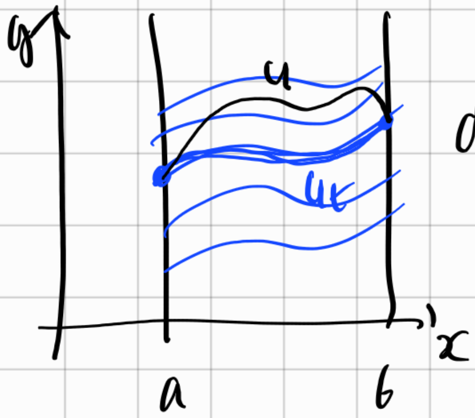
$$+\frac{\partial L}{\partial z}(t,u(t),\bar z(t,u(t)))\,u'(t)\Big]dt$$

$$= \int_a^b M(x,u(x),u'(x))\,dx \qquad \gamma(t) = \begin{pmatrix} x(t) \\ y(t)\end{pmatrix}$$

$$\omega(x,y) = \boxed{\alpha(x,y)\,dx + \beta(x,y)\,dy} \qquad \begin{cases} dx = x'(t)\,dt \\ dy = y'(t)\,dt\end{cases}$$

$$\int_\gamma \omega = \int\left[\alpha(\gamma(t))\,x'(t) + \beta(\gamma(t))\,y'(t)\right]dt$$

Ci siamo ricondotti a dimostrare che ω è esatta.



$$\mathcal{M}(u) - \mathcal{M}(u_t) = \int_{\gamma_u}\omega - \int_{\gamma_{u_t}}\omega$$

$$= \oint \omega = 0$$

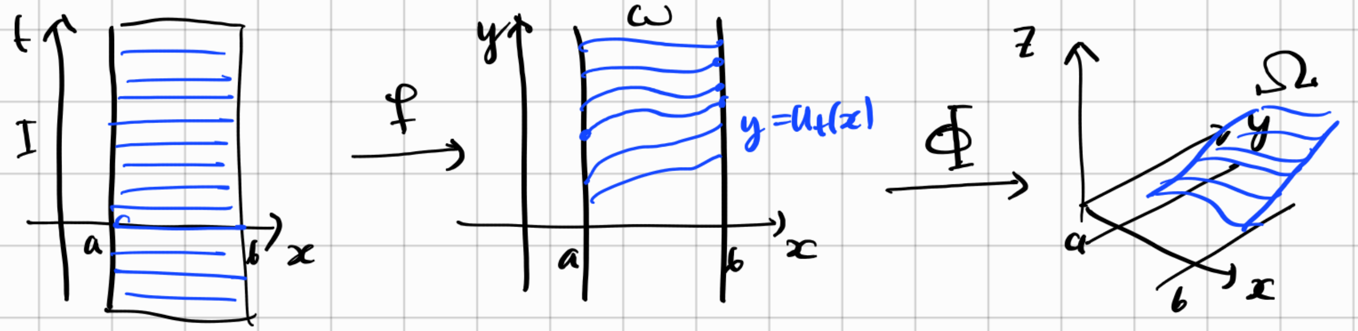
se ω è esatta

Dimostriamo che ω è chiusa.
visto che G è omeomorfo a $[a,b]\times I$, G è semplicemente connesso $\Rightarrow$ ω chiusa $\Leftrightarrow$ ω esatta.

$$\omega = \alpha(x,y)\,dx + \beta(x,y)\,dy$$

ω è chiusa se $\boxed{\dfrac{\partial \alpha}{\partial y} = \dfrac{\partial \beta}{\partial x}}$

$$\left[\ \text{Se } \omega = dS \qquad \alpha = \frac{\partial S}{\partial x},\ \beta = \frac{\partial S}{\partial y} \Rightarrow \frac{\partial^2 S}{\partial x \partial y} = \frac{\partial^2 S}{\partial y \partial x}\ \right]$$



$$f(x,t) = (x, u_t(x)) \qquad y = u_t(x)$$

$$\Phi(x,y) = (x, y, \bar{z}(x,y))$$

$$\omega(x,y) = \underbrace{\left[L(x,y,\bar{z}(x,y)) - \bar{z}(x,y)\cdot \frac{\partial L}{\partial z}(x,y,\bar{z}(x,y))\right]}_{\alpha(x,y)} dx + \underbrace{\frac{\partial L}{\partial z}(x,y,\bar{z}(x,y))}_{\beta(x,y)}\, dy$$

$$\omega = \alpha\, dx + \beta\, dy \qquad \omega \overset{?}{=} \Phi_* \Omega$$

$$\Omega = \left[L(x,y,z) - z\frac{\partial L}{\partial z}(x,y,z)\right]dx + \frac{\partial L}{\partial z}(x,y,z)\,dy + 0\cdot dz$$

$$\Omega = \left(L - z\cdot\frac{\partial L}{\partial z}\right)dx + \frac{\partial L}{\partial z}\cdot dy$$ Forma di Beltrami

Ricordo:

$$\boxed{\omega = \Phi_* \Omega \quad \text{se} \quad \omega(x,y)[v] = \Omega(\Phi(x,y))\,[D\Phi[v]]}$$

$$\Phi(x,y) = \begin{pmatrix} x \\ y \\ \bar{z}(x,y) \end{pmatrix} \qquad \begin{aligned} dx &= dx \\ dy &= dy \\ dz &= \frac{\partial \bar{z}}{\partial x}dx + \frac{\partial \bar{z}}{\partial y}dy. \end{aligned}$$

Idea: dimostrare che $f^*\omega$ è chiusa siccome f è un diffeomorfismo

ω chiusa $\Leftrightarrow$ $f_*\omega$ chiusa

$$f_*\omega = \underbrace{(\phi\circ f)_*(\Omega)} = f_*(\Phi_*(\Omega))$$

$$f_*\omega = (\Phi\circ f)_*(\Omega)$$

$$f(x,t) = (x, u_t(x))$$
$$\phi(x,y) = (x, y, \overline{z}(x,y))$$
$$\overline{z}(x, u_t(x)) = u_t'(x)$$

$$\phi(f(x,t)) = \phi(x, u_t(x)) = (x, u_t(x), u_t'(x))$$

$$\Omega = \left(L - z\cdot\frac{\partial L}{\partial z}\right)dx + \frac{\partial L}{\partial z}\, dy$$

$$f_*\omega = (\phi\circ f)_*(\Omega)$$
$$= \left[L(x,u_t(x),u_t'(x)) - \cancel{u_t'(x)\frac{\partial L}{\partial z}(x,u_t(x),u_t'(x))}\right]dx$$
$$+ \frac{\partial L}{\partial z}(x,u_t(x),u_t'(x))\cdot\left(\cancel{u_t'(x)dx} + \frac{\partial g(x,t)}{\partial t}.dt\right)$$

$$\boxed{\begin{gathered} y = u_t(x) =: g(x,t) \\ dy = \underbrace{\frac{\partial g}{\partial x}}_{\substack{\| \\ u_t'(x)}} dx + \frac{\partial g}{\partial t}.dt \end{gathered}}$$

$$= \underbrace{L(x,u_t(x),u_t'(x))}\,dx + \underbrace{\frac{\partial L}{\partial z}(x,u_t(x),u_t'(x))\,\frac{\partial g}{\partial t}.dt}$$

è chiusa?

$$d f_*\omega = \frac{d}{dt}L(x,u_t(x),\underbrace{u_t'(x)}) - \frac{d}{dx}\left(\frac{\partial L}{\partial z}(x,u_t(x),u_t'(x))\cdot\frac{\partial g(x,t)}{\partial t}\right) =$$

$$\left(u_t \text{ soddisfa E.L.}: \quad \frac{d}{dx}\frac{\partial L}{\partial z}(x,u_t(x),u_t'(x)) = \frac{\partial L}{\partial y}(x,u_t(x),u_t'(x))\right)$$

$$= \frac{\partial L}{\partial y}\cdot\frac{\partial g(x,t)}{\partial t} + \cancel{\frac{\partial L}{\partial z}\cdot\frac{\partial^2 g(x,t)}{\partial x\partial t}} - \left(\frac{d}{dx}\left(\frac{\partial L}{\partial z}\right)\right)\cdot\frac{\partial g(x,t)}{\partial t} - \cancel{\frac{\partial L}{\partial z}\cdot\frac{\partial^2 g(x,t)}{\partial t\partial x}}$$

$$= \underbrace{\cancel{\left[\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial z}\right]}}_{EL} \cdot \frac{\partial g}{\partial t} = 0 \qquad \square$$

Esempio $\quad L(x,y,z) = \frac{1}{2}z^2 - \frac{1}{2}y^2 \qquad$ L è convessa in z $\forall y$.

$$\mathcal{L}(u) = \frac{1}{2}\int_0^T \left[(u'(x))^2 - (u(x))^2\right] dx$$

$$\begin{cases} u(a) = 0 \\ u(b) = 0 \end{cases} \qquad \mathcal{L}(u) \to \min.$$

$$\frac{\partial L}{\partial y} = -y \qquad \frac{\partial L}{\partial z} = z \qquad \text{E.L:} \quad \frac{\partial L}{\partial y} = \frac{d}{dx}\frac{\partial L}{\partial z}$$

$$-u(x) = \frac{d}{dx}u'(x) = u''(x)$$

E.L.: $\quad u'' + u = 0 \quad$ eq. lineare, II ordine, coefficienti costanti omogenea.

$$P(\lambda) = \lambda^2 + 1 \qquad \lambda_{12} = \pm i \qquad C = \sqrt{A^2 + B^2}$$

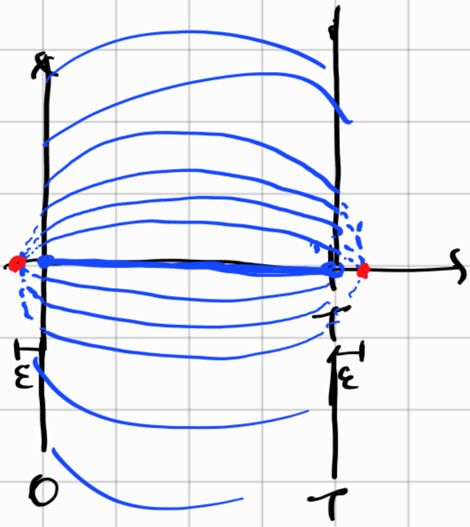
$$u(x) = A \cdot \sin x + B\cos x = C \cdot \sin(x - x_0)$$

$$u(0) = 0 \implies B = 0$$

$$u(T) = 0 \implies A \cdot \sin(T) = 0$$

se $T \neq k \cdot \pi$ allora $A = 0$, $u = 0$

Supponiamo $0 < T < \pi$. Allora $u = 0$ è l'unica soluzione di E.L. tale che $u(0) = 0$, $u(T) = 0$



$\pi > T$

$$u_t(x) = t \cdot \sin(x + \varepsilon)$$

con $\varepsilon = \frac{\pi - T}{2}$

$f(x,t) = (x, u_t(x))$ è un diffeomorfismo su tutto $[0,T] \times \mathbb{R}$.

E' una calibrazione. Ogni u_t è minimo di $\mathcal{L}$ con il suo dato al bordo $\Rightarrow$ $u_0(x) = 0$ è minimo con dato al bordo $u(0)=0$, $u(T)=0$.

$$\mathcal{L}(u) \geq \mathcal{L}(u_0) = 0 \Rightarrow \int_0^T (u')^2 \geq \int_0^T u^2$$

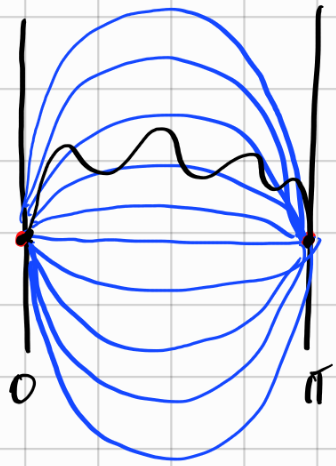
$\forall u \in C_0^1([0,T])$, se $T<\pi$, $\|u\|_{L^2} \leq \|u'\|_{L^2}$

Se $T=\pi$

VERIFICARE CHE:

$$\boxed{\mathcal{L}(u_t) = 0}$$

u_t è minimo $\forall t$ con $u(0)=0$, $u(\pi)=0$



$u_t(x) = t \cdot \sin x$

$f(x,t) = u_t(x)$ è un diffeomorfismo solo all'interno: $(a,b)\times\mathbb{R}$ ma tutto funziona ugualmente.

Se $T>\pi$ verifichiamo che $\mathcal{L}$ non ha minimo, in quanto $\inf \mathcal{L} = -\infty$.

$$\mathcal{L}(t\cdot u) = \frac{1}{2}\int (tu')^2 + (tu)^2 = t^2 \mathcal{L}(u)$$

Basta trovare u tale che $\mathcal{L}(u)<0$ $\quad \mathcal{L}(tu) \longrightarrow -\infty$ per $t\to+\infty$.

$u(x) = \sin\left(\frac{\pi x}{T}\right) \qquad u(0)=0,\ u(T)=0$

$$u'(x) = \frac{\pi}{T}\cos\left(\frac{\pi x}{T}\right) \qquad \mathcal{L}(u) = \frac{1}{2}\int_0^T \left[\frac{\pi^2}{T^2}\cos^2\left(\frac{\pi x}{T}\right) - \sin^2\left(\frac{\pi x}{T}\right)\right]dx$$

$$= \frac{1}{2}\left[\frac{\pi^2}{T^2}\frac{1}{2} - \frac{1}{2}\right]$$

$$= \frac{1}{4}\left[\frac{\pi^2}{T^2} - 1\right] < 0 \quad \text{se } T>\pi$$

$\square$