ON THE TOTAL SURFACE AREA OF POTATO PACKINGS

MATTEO NOVAGA, EMANUELE PAOLINI, AND EUGENE STEPANOV

Abstract. We prove that if we fill without gaps a bag with infinitely many potatoes, in such a way that they touch each other in few points, then the total surface area of the potatoes must be infinite. In this context potatoes are measurable subsets of the Euclidean space, the bag is any open set of the same space. As we show, this result also holds in the general context of doubling (even locally) metric measure spaces satisfying Poincaré inequality, in particular in smooth Riemannian manifolds and even in some sub-Riemannian spaces.

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Introduction

In this note we prove the following curious fact: if we fill without gaps a bag (modeled, say, by an arbitrary open connected subset of $\mathbb{R}^d$) with potatoes (modeled, in this case, by open sets of finite perimeter), in such a way that the potatoes touch each other in a single point (more generally in a set of zero surface measure), see Figure 1, then the total surface area of the potatoes is infinite. We show that this very simple result is in fact true for a fairly large class of metric measure spaces, where the perimeter of a set can be naturally defined.

This result implies in particular that the residual set (the complement of the union of the potatoes) has Hausdorff dimension at least $d-1$. Theorem 1.4(ii) from [14] shows that this statement is sharp: in fact it proves the existence of a packing of a planar convex set by planar strictly convex sets for which the dimension of the residual set is exactly 1.

In this way we generalise the results of [14], dedicated to packing of convex sets, which in their turn generalise those from the series of papers [12, 10, 11, 13] and [15] dedicated to packings of spheres.

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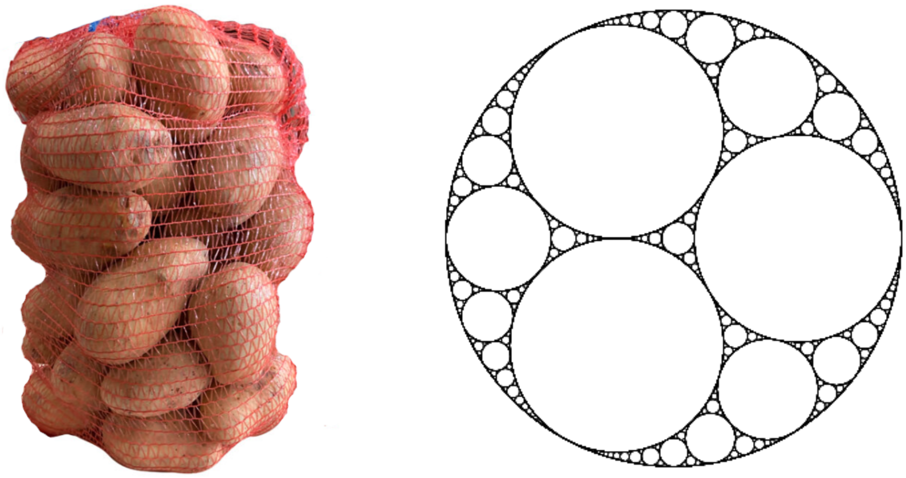

FIGURE 1. On the left-hand side: a potato bag. On the right-hand side: an Apollonian gasket.

1. Notation and preliminary results

We will consider here metric measure spaces $(X, \mathrm{d}, \mathfrak{m})$ where X is a nonempty set equipped with a distance d and a possibly σ-finite Borel measure $\mathfrak{m}$ with $\mathfrak{m}(X) \neq 0$. We say that a function F defined on the borel sets of a metric space X with values in $[0, +\infty]$ is a *perimeter-like evaluation* if it satisfies the following properties:

- (0) $F(\emptyset) = 0$;
- (T) $F(A) \geq \limsup_n F(A_n)$ whenever $A_n \subset A$ and $F(A \setminus A_n) \to 0$;
- (C) $F(X \setminus A) = F(A)$;
- (L) $F(A) \leq \liminf_n F(A_n)$ if $\lim_n \mathfrak{m}(A_n \triangle A) = 0$ (one says that F is lower semicontinuous with respect to $L^1(X, \mathfrak{m})$ convergence of sets);
- (Z) if $\mathfrak{m}(A \triangle B) = 0$ then $F(A) = F(B)$.

Lemma 1.1. *The property (T) is valid if*

(T') *there exists a $c > 0$ such that $F(A) \geq F(A \setminus B) - cF(B)$ whenever $B \subset A$.*

Proof. Just notice that

$$\begin{aligned} F(A) &= F(A_n \cup (A \setminus A_n)) \\ &\geq F(A_n) - cF(A \setminus A_n) \qquad \text{by } (T') \end{aligned}$$

and if $F(A \setminus A_n) \to 0$, then we have (T). □

For a fairly general class of metric measure spaces, namely those with a doubling measure and satisfying the Poincaré inequality, a perimeter functional has been defined in [16, 3, 4]. We refer to the monograph [8] and references therein for a detailed discussion of such spaces. Here we just recall the basic definitions. A measure $\mathfrak{m}$ is said *doubling*, provided there is a $C_D \geq 1$ such that

$$\mathfrak{m}(B_{2r}(x)) \leq C_D \mathfrak{m}(B_r(x)), \qquad \text{for all } x \in X, r \in (0, R).$$

We will shortly say that $(X, \mathrm{d}, \mathfrak{m})$ is a doubling space.

We say that a metric measure space $(X, \mathrm{d}, \mathfrak{m})$ supports a *weak* $(1,1)$ *Poincaré* inequality, provided there is $\tau_P \geq 1$, and a constant $C_P > 0$ such that, for any locally Lipschitz function $f\colon X \to \mathbb{R}$, one has

$$\fint_{B_r(x)} |f - f_{x,r}| \, d\mathfrak{m} \leq C_P r \fint_{B_{\tau_P r}(x)} \mathrm{Lip}(f) \, d\mathfrak{m}, \qquad \text{for all } x \in X, r > 0,$$

where $f_{x,r} := \fint_{B_r(x)} f \, d\mathfrak{m}$ and $Lip(f)$ stands for the Lipschitz constant of f over $B_{\tau_P r}(x)$. Actually, this is usually formulated with *upper gradients* (see [7]), rather than with local Lipschitz constant. However, if $(X, \mathrm{d}, \mathfrak{m})$ is doubling, the two formulations are equivalent [9].

Definition 1.2. A PI-space is a metric measure space $(X, \mathrm{d}, \mathfrak{m})$ that is doubling and supports a weak $(1,1)$-local Poincaré inequality.

The perimeter functional in a PI-space has been defined in [16, 3] (see also [4]) as a natural generalization of the classical Euclidean perimeter in $\mathbb{R}^d$ (see [6]). Note that, like in the classical Euclidean case $\mathbb{R}^d$ where the perimeter is the $(d-1)$-dimensional Hausdorff measure of the essential boundary, also in PI-spaces the perimeter of a set E is a measure of the essential boundary $\partial^e E$ of E, defined in [3], and is absolutely continuous with respect to the so-called codimension one Hausdorff measure $\mathcal{H}^{-1}$, with a Borel density $\theta_E\colon X \to [\alpha, \beta]$ where α and β are two positive constants depending only on the constants C_D, C_P, τ_P of the space [4]. In particular, one has the integral representation

$$P(E) = \int_{\partial^e E} \theta_E(x) \, d\mathcal{H}^{-1}(x).$$

If, moreover,

$$\theta_E = \theta_F \qquad \mathcal{H}^{-1}\text{-a.e. on } \partial^e E \cap \partial^e F,$$

whenever $E, F \subset X$ are two sets of finite perimeter satisfying $E \subset F$, then the PI-space $(X, \mathrm{d}, \mathfrak{m})$ is called *isotropic*. We refer to [4, 5] where this notion has been further studied. It is important to know that this includes a many classical examples, like for instance

1. the Euclidean space $X = \mathbb{R}^d$ (or X open subset of $\mathbb{R}^d$) with the Lebesgue measure (in this case $\mathcal{H}^{-1}$ is the classical $(d-1)$-dimensional Hausdorff measure and $\partial^e E$ is equivalent to the reduced boundary of E as defined by De Giorgi),
2. X a finite dimensional space equipped with any anisotropic norm and the Lebesgue measure, or even some of the so-called *RCD* metric measure spaces,
3. X a Heisenberg group of any dimension, or even some of the more general Carnot groups.

Lemma 1.3. *Let* $(X, \mathrm{d}, \mathfrak{m})$ *be an isotropic PI space. Then the perimeter functional* P *is a perimeter-like evaluation, i.e. it satisfies all the properties (0), (T), (C), (L), and (Z).*

Proof. Property (0) follows from $\partial^e \emptyset = \emptyset$. Proposition 1.7 (i), (ii), and (vi) in [4] shows (Z), (L), and (C) respectively.

Let us prove (T') which by Lemma 1.1 implies (T). By theorem 1.23 in [4] we have

$$P(E) = \int_{\partial^e E} \theta_E(x) \, d\mathcal{H}^{-1}(x)$$

and $\theta_E(x) \in [\alpha, \beta]$ where $0 < \alpha \leq \beta$ are constants depending only on the metric measure space $(X, \mathrm{d}, \mathfrak{m})$ but not on the set E. Then, for Borel sets $B \subset A \subset X$, one has

$$\begin{aligned}\partial^e(A \setminus B) &= \emptyset \cup \partial^e(A \setminus B) = \partial^e X \cup \partial^e(A \setminus B) \\ &= \partial^e(A \cup B^c) \cup \partial^e(A \cap B^c) \\ &\subset \partial^e A \cup \partial^e B^c && \text{by proposition 1.16(ii) in [4]} \\ &= \partial^e A \cup \partial^e B && \text{by proposition 1.16(i) in [4].}\end{aligned}$$

Notice also that $\theta_{A\setminus B}(x) = \theta_A(x)$ for $\mathcal{H}^{-1}$-a.e. $x \in \partial^e A \cap \partial^e(A \setminus B)$ since X is isotropic. Hence

$$\begin{aligned}P(A \setminus B) &= \int_{\partial^e(A\setminus B)} \theta_{A\setminus B}(x)\, d\mathcal{H}^{-1}(x) \\ &= \int_{\partial^e A\cap\partial^e(A\setminus B)} \theta_{A\setminus B}(x)\, d\mathcal{H}^{-1}(x) + \int_{(\partial^e B\setminus\partial^e A)\cap\partial^e(A\setminus B)} \theta_{A\setminus B}(x)\, d\mathcal{H}^{-1}(x) \\ &\leq \int_{\partial^e A\cap\partial^e(A\setminus B)} \theta_A(x)\, d\mathcal{H}^{-1}(x) + \int_{\partial^e B} \beta\, d\mathcal{H}^{-1}(x) \\ &\leq P(A) + \frac{\beta}{\alpha} P(B).\end{aligned}$$

showing (T') and thus concluding the proof. □

2. Main result

We start with the following auxiliary statement.

Proposition 2.1. *If F satisfies properties (0), (C), and the family of sets E_k, $k \in \mathbb{N}$ is such that $\bigcup E_i = X$, and*

$$F\left(\bigcup_j E_j\right) \geq \sum_j F(E_j)$$

then $F(E_i) = 0$ for all $i \in \mathbb{N}$.

If, additionally, F satisfies (Z) then the same holds true if $\mathfrak{m}(X \setminus \bigcup E_i) = 0$ rather than $\bigcup E_i = X$.

Proof. One has

$$\begin{aligned}0 &= F(\emptyset) && \text{by (0)} \\ &= F(X) && \text{by (C)} \\ &= F\left(\bigcup E_i\right) && \text{by assumption} \\ &\geq \sum_i F(E_i) && \text{by assumption}\end{aligned}$$

hence $F(E_i) = 0$ for all $i \in \mathbb{N}$.

If, additionally, F satisfies (Z), and $\mathfrak{m}(X \setminus \bigcup E_i) = 0$ one has

$$\begin{aligned} 0 = F(\emptyset) &= F(X) \\ &= F\left(X \setminus \bigcup E_i\right) && \text{by (Z)} \\ &= F\left(\bigcup E_j\right) && \text{by (C)} \\ &\geq \sum F(E_j) && \text{by assumption} \end{aligned}$$

so that $F(E_j) = 0$ for all j. □

The Theorem below is the main tecnical result which will be further translated into corollaries adapted to applications.

Theorem 2.2. *If F satisfies properties (0), (C), (T), (L) and the family of sets E_k, $k \in \mathbb{N}$ is such that $F(E_i \cup E_j) = F(E_i) + F(E_j)$ for $i \neq j$, $\bigcup_{i=0}^{+\infty} E_i = X$, and $\mathfrak{m}\left(\bigcup_{i=1}^{+\infty} E_i\right) < +\infty$, then either $\sum_{i=0}^{+\infty} F(E_i) = +\infty$ or $F(E_i) = 0$ for all $i \in \mathbb{N}$.*

If, additionally, F satisfies (Z) then the same holds true if $\mathfrak{m}(X \setminus \bigcup_{i=0}^{+\infty} E_i) = 0$ for all $i \neq j$, in place of $\bigcup E_i = X$.

Proof. Let

$$T_n := \bigcup_{i=n+1}^{+\infty} E_j, \qquad T_n^m := \bigcup_{i=n+1}^{m} E_j.$$

Clearly $T_n^m \nearrow T_n$ as $m \to +\infty$, hence $\lim_m \mathfrak{m}(T_n^m) = \mathfrak{m}(T_n)$. Since $\mathfrak{m}(T_n) \leq \mathfrak{m}\left(\bigcup_{i=1}^{+\infty} E_i\right) < +\infty$ then we have that $\mathfrak{m}(T_n^m \triangle T_n) \to 0$ as $m \to +\infty$. Therefore

$$\begin{aligned} F(T_n) &\leq \liminf_m F(T_n^m) && \text{by (L)} \\ &= \liminf_m \sum_{i=n+1}^{m} F(E_i) && \text{by assumption} \\ &= \sum_{i=n+1}^{+\infty} F(E_i). \end{aligned} \tag{1}$$

If $\sum_{i=1}^{+\infty} F(E_i) = +\infty$ the proof is concluded. Otherwise the righthand side of (1) is the remainder of a convergent number series and hence $F(T_n) \to 0$ as $n \to +\infty$. We have then

$$\begin{aligned} F\left(\bigcup_{i=0}^{+\infty} E_i\right) &\geq \limsup_n F\left(\bigcup_{i=0}^{n} E_i\right) && \text{by (T)} \\ &= \lim_n \sum_{i=0}^{n} F(E_i) && \text{by assumption} \\ &= \sum_{i=0}^{+\infty} F(E_i). \end{aligned}$$

The proof is then concluded using the previous proposition. □

3. Some applications

Combining Theorem 2.2 with Lemma 1.3 we obtain the following result.

Theorem 3.1. *Let $(X, \mathrm{d}, \mathfrak{m})$ be an isotropic PI space. Let E_k, $k \in \mathbb{N}$, be a sequence of Borel subsets of X such that*

(i) $\mathfrak{m}(E_k \cap E_j) = 0$ *for* $k \neq j$*;*
(ii) $\mathfrak{m}(X \setminus \bigcup E_k) = 0$*;*
(iii) $\mathcal{H}^{-1}(\partial^e E_k \cap \partial^e E_j) = 0$ *for all* $k \neq j$*.*
(iv) $\mathfrak{m}(E_0) > 0$ *and* $\mathfrak{m}(E_1) > 0$*.*

Then

$$\sum_k P(E_k) = +\infty.$$

Proof. Conditions (i) and (iii) imply

$$P(E_k \cup E_j) = P(E_k) + P(E_j)$$

in view of Lemma 2.3(ii) in [4]. The assumptions of Theorem 2.2 are therefore satisfied by P in view of Lemma 1.3. Hence by Theorem 2.2 we conclude that either $\sum P(E_k) = +\infty$ or $P(E_k) = 0$ for all k. But the latter option is excluded by (iv) in view of the relative isoperimetric inequality for PI spaces, provided in theorem 1.17 of [4]. □

Corollary 3.2. *Let $\mathfrak{m}$ be the Lebesgue measure on $\mathbb{R}^d$. Let $\mathcal{F}$ be a family of at least two measurable nonempty subsets of $\mathbb{R}^d$, each with positive volume $\mathfrak{m}$ and kissing each other on sets of zero $\mathcal{H}^{d-1}$ measure, i.e.*

$$\mathcal{H}^{d-1}(\bar{A} \cap \bar{B}) = 0, \qquad \text{for every } A, B \in \mathcal{F}, A \neq B. \tag{2}$$

In particular this assumption is satisfied if the sets in $\mathcal{F}$ are strictly convex.

Let $B \subset \mathbb{R}^d$ be an open ball. If $\sum_{E \in \mathcal{F}} \mathcal{H}^{d-1}(\partial E \cap B) < +\infty$, then either there exists a set $E \in \mathcal{F}$ such that

$$\mathfrak{m}(B \setminus E) = 0$$

or

$$\mathfrak{m}\left(B \setminus \bigcup_{E \in \mathcal{F}} E\right) > 0,$$

i.e. there is a subset of B with positive measure which is not covered by the union of $\mathcal{F}$.

In particular if $\Omega \subset \mathbb{R}^d$ is an open set (a "bag of potatos"), and all $E \in \mathcal{F}$ are open nonempty subsets of Ω, regular in the sense that they coincide with the interior of their closure, satisfying (2) and $\mathfrak{m}(\Omega \setminus \bigcup_{E \in \mathcal{F}} E) = 0$, then given any $x \in \Omega \cap \bigcup_{E \in \mathcal{F}} \partial E$, and B an open ball centered at x, one has

$$\sum_{E \in \mathcal{F}} \mathcal{H}^{d-1}(\partial E \cap B) = +\infty. \tag{3}$$

If, moreover, Ω is also connected, this implies in particular

$$\sum_{E \in \mathcal{F}} \mathcal{H}^{d-1}(\partial E \cap \Omega) = +\infty. \tag{4}$$

If $d = 2$ and all $E \in \mathcal{F}$ are convex, this also implies

$$\sum_{E \in \mathcal{F}} \mathrm{diam} E = +\infty. \tag{5}$$

Proof. We apply Theorem 3.1 with $X := B$, equipped with the Euclidean distance, and P is the usual Euclidean (Caccioppoli) perimeter relative to B. Clearly, the doubling condition holds for B and, since B is connected, the Poincaré inequality is satisfied and $X = B$ is an isotropic PI space. Then, the De Giorgi reduced boundary $\partial^* E$ of a Borel set $E \subset \mathbb{R}^d$ satisfies $\partial^* E \subset \partial^e E$ and $\mathcal{H}^{d-1}(\partial^e E) = \mathcal{H}^{d-1}(\partial^* E)$. Notice that $\mathcal{H}^{d-1}$ coincides with the spherical Hausdorff measure $\mathcal{S}^{d-1}$ on rectifiable sets, and (see the Example "weighted spaces" in Section 7 of [3]) $\mathcal{H}^{-1}$ coincides, up to a multiplicative constant, with $\mathcal{S}^{d-1}$.

If there exists $E \in \mathcal{F}$ such that $\mathfrak{m}(B \setminus E) = 0$ or, if $\mathfrak{m}(B \setminus \bigcup_{E\in\mathcal{F}} E) > 0$ there is nothing to prove. Otherwise there should be at least two different sets E_0, E_1 in $\mathcal{F}$ such that $\mathfrak{m}(B \cap E_0) > 0$ and $\mathfrak{m}(B \cap E_1) > 0$. Enumerate now all the sets of $\mathcal{F}$ as E_k, $k \in \mathbb{N}$ (clearly $\mathcal{F}$ is at most countable since each set is assumed to have positive measure, and in the case that $\mathcal{F}$ is finite we can complete the sequence with empty sets). Notice that $\mathcal{H}^{-1}(\partial^e E_k \cap \partial^e E_j) = 0$ for all $k \neq j$ and $\mathfrak{m}(E_k \cap E_j) = 0$ since $\mathcal{H}^{d-1}(\bar{E}_k \cap \bar{E}_j) = 0$.

Therefore, we can apply Theorem 3.1 to the sequence $E_k \cap B$ to get $\sum_k P(E_k, B) = +\infty$ hence

$$\sum_k \mathcal{H}^{d-1}(\partial E_k \cap B) \geq \sum_k \mathcal{H}^{d-1}(\partial^* E_k \cap B) = \sum_k P(E_k, B) = +\infty,$$

showing the first claim.

In the particular case when $\mathfrak{m}(\Omega \setminus \bigcup_{E\in\mathcal{F}} E) = 0$ for some open $\Omega \subset \mathbb{R}^d$, taking x and B as in the statement, we consider a ball $B' \subset B$ centered at x such that $B' \subset \Omega$. Then, for every $E \in \mathcal{F}$ either $E \cap B' = \emptyset$ or $E \cap B' \neq \emptyset$. In the latter case one has $\partial E \cap B' \neq \emptyset$ since otherwise we would have $B' \subset E$ which contradicts the choice of the center x and the fact that all the sets in $\mathcal{F}$ are disjoint. The regularity assumption implies that the interior of $B' \setminus E$ is nonempty, hence $\mathfrak{m}(B' \setminus E) > 0$. Therefore we have shown that $\mathfrak{m}(B' \setminus E) > 0$ for every $E \in \mathcal{F}$ and hence by the previous claim with B' in place of B, one has

$$\sum_{E\in\mathcal{F}} \mathcal{H}^{d-1}(\partial E \cap B) \geq \sum_{E\in\mathcal{F}} \mathcal{H}^{d-1}(\partial E \cap B') = +\infty$$

as claimed.

If Ω is an open and connected set, and $\mathcal{F}$ has at least two elements, we obtain that $\bigcup_{E\in\mathcal{F}} \partial E \cap \Omega$ is not empty. Finally, from (3) one has

$$\sum_{E\in\mathcal{F}} \mathcal{H}^{d-1}(\partial E) = +\infty,$$

and hence, if $d = 2$ and all $E \in \mathcal{F}$ are convex, then $\text{diam} E \leq \mathcal{H}^1(\partial E)$ so that also (5) follows.

□

Remark 3.3. It is clear from the proof that the statement of Corollary 3.2 is valid under the slightly weaker assumption that $\mathcal{H}^{d-1}(\partial^* A \cap \partial^* B) = 0$ for all $A, B \in \mathcal{F}$ instead of $\mathcal{H}^{d-1}(\bar{A} \cap \bar{B}) = 0$, where ∂^* stands for the reduced boundary in the sense of De Giorgi.

A possible alternative proof of the corollary for the family $\mathcal{F}$ of strictly convex sets could proceed as follows:

- For the planar case $d = 2$ one uses the coarea inequality to state that if the total perimeter of the sets in the family is finite then almost every line (in fact every except a countable number of lines), say horizontal, intersects the boundary of the sets in a set of finite $\mathcal{H}^0$ measure (i.e. in a finite set). On the other hand if the sets of $\mathcal{F}$ are assumed to be strictly convex then the lines intersecting only a finite number of sets in the family should be at most countable: in fact, the intersection of a convex set with a line is a line segment, and hence if the line intersects only a finite number of sets, then each point of its intersection with the boundary of some set of $\mathcal{F}$ is a point of intersection between two different sets of $\mathcal{F}$, which are countably many in total. This contradiction shows that the total perimeter of the sets in the family is infinite for planar packing of strictly convex sets (even in the case when the ambient set is not convex).
- For the general space dimension $d \geq 2$, again assuming that the total perimeter of the packing is convex, we have again by coarea inequality that the intersection of almost every hyperplane, say, horizontal, with the boundary of the sets in the family, has finite Hausdorff measure $\mathcal{H}^{d-2}$. If the ambient set is convex, then inside every such plane we have a packing of a convex set by strictly convex sets. Proceeding by backward induction on the dimension we arrive at a contradiction. Convexity of the ambient set is clearly essential for the argument to work in the case $d > 2$.

Corollary 3.4. *Corollary 3.2 is valid in any metric measure space* $(X, \mathrm{d}, \mathfrak{m})$ *satisfying the Poincaré inequality where for every* $x \in X$ *there is an open ball* $U \subset X$ *containing* x *such that* $(U, \mathrm{d}, \mathfrak{m})$ *is doubling. In particular it is valid in any* C^2 *smooth Riemannian manifold.*

Proof. It is enough to rewrite word-to-word the proof of Corollary 3.2 with balls $B \subset U$. In the case when X a C^2 smooth Riemannian manifold it is enough to note that over every ball U the curvature of X is bounded hence $(U, \mathrm{d}, \mathfrak{m})$ is doubling (in fact it is enough for this purpose that the curvature be bounded from below). □

References

1. L. Ambrosio, *Some fine properties of sets of finite perimeter in Ahlfors regular metric measure spaces*, Adv. Math. **159** (2001), no. 1, 51–67.
2. ———, *Fine properties of sets of finite perimeter in doubling metric measure spaces*, Set-Valued Anal. **10** (2002), no. 2-3, 111–128.
3. L. Ambrosio, M. Miranda, Jr., and D. Pallara, *Special functions of bounded variation in doubling metric measure spaces*, Calculus of variations: topics from the mathematical heritage of E. De Giorgi, Quad. Mat., vol. 14, Dept. Math., Seconda Univ. Napoli, Caserta, 2004, pp. 1–45.
4. P. Bonicatto, E. Pasqualetto, and T. Rajala, *Indecomposable sets of finite perimeter in doubling metric measure spaces*, Calc. Var. Partial Differential Equations **59** (2020), no. 2, Paper No. 63, 39. MR 4073209
5. C. Brena, F. Nobili, and E. Pasqualetto, *Maps of bounded variation from PI spaces to metric spaces*, Preprint, ArXiv:2306.00768 (2023), 2023.
6. E. Giusti, *Minimal surfaces and functions of bounded variation*, Monographs in Mathematics, vol. 80, Birkhäuser, 1984.
7. J. Heinonen and P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry*, Acta Math. **181** (1998), no. 1, 1–61.
8. J. Heinonen, P. Koskela, N. Shanmugalingam, and J. T. Tyson, *Sobolev spaces on metric measure spaces. An approach based on upper gradients*, New Math. Monogr., vol. 27, Cambridge: Cambridge University Press, 2015.

9. S. Keith, *Modulus and the Poincaré inequality on metric measure spaces*, Math. Z. **245** (2003), no. 2, 255–292.
10. D. G. Larman, *An asymptotic bound for the residual area of a packing of discs*, Proc. Cambridge Philos. Soc. **62** (1966), 699–704.
11. ———, *A note on the Besicovitch dimension of the closest packing of spheres in R_n*, Proc. Cambridge Philos. Soc. **62** (1966), 193–195.
12. ———, *On the exponent of convergence of a packing of spheres*, Mathematika **13** (1966), no. 1, 57–59.
13. ———, *On packings of unequal spheres in R_n*, Canadian J. Math. **20** (1968), 967–969.
14. S. Maio and D. Ntalampekos, *On the Hausdorff dimension of the residual set of a packing by smooth curves*, Journal of the London Mathematical Society **105** (2022), no. 3, 1752–1786.
15. A. S. Mikhailov and V. S. Mikhailov, *Remark on covering theorem*, Journal of Mathematical Sciences **112** (2002), 4024–4028.
16. M. Miranda, *Functions of bounded variation on "good" metric spaces*, J. Math. Pures Appl. (9) **82** (2003), no. 8, 975–1004.

(Matteo Novaga, Emanuele Paolini, Eugene Stepanov) Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, I-56127, Pisa

(Eugene Stepanov) St.Petersburg Branch of the Steklov Mathematical Institute of the Russian Academy of Sciences, St.Petersburg, Russia and Faculty of Mathematics, Higher School of Economics, Moscow

Email address, Matteo Novaga: matteo.novaga@unipi.it

Email address, Emanuele Paolini: emanuele.paolini@unipi.it

Email address, Eugene Stepanov: stepanov.eugene@gmail.com