OPTIMAL SETS FOR A GEOMETRIC OSCILLATION ENERGY

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ABSTRACT. We investigate the nonlocal energy corresponding to the p-oscillation of the unit normal vector for hypersurfaces, or the unit tangent vector for curves. The energy satisfies geometric inequalities with optimal constants $c(n,p)$ and $C(n,p)$ which are determined by a variational problem over the probability measures on the sphere. The extremal measures for such problem depend critically on the value of p. We prove existence of optimal sets for this energy under perimeter and volume constraint, and characterize their shape.

CONTENTS

1. INTRODUCTION

Nonlocal geometric functionals have attracted considerable attention in recent years due to their rich mathematical structure and their appearance in physical models, such as in the theory of membranes and vesicles. A typical example of nonlocal energies is the fractional perimeter, introduced in [7] as a model of phase transitions with long-range interactions. Since then, minimization problems involving the fractional perimeter have been studied by many authors (see for instance [9, 19] and references therein). Another example is the nonlocal Willmore energy or its variants, which involve double integrals of a kernel depending on both position and normal vector (see for instance [3, 16, 8, 4] for further details).

In this paper, we consider a nonlocal geometric energy that depends on the normal vectors. More precisely, given $p > 0$ and a set $\Omega \subset \mathbb{R}^n$ of finite volume and finite perimeter, we consider the energy

$$\mathrm{E}_p(\Omega) := \iint_{\partial^*\Omega \times \partial^*\Omega} |\nu_\Omega(x) - \nu_\Omega(y)|^p \, d\mathcal{H}^{n-1}(x) \, d\mathcal{H}^{n-1}(y) \tag{1}$$

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where $\partial^*\Omega$ denotes the reduced boundary of Ω and $\nu_\Omega(x)$ the measure-theoretic outer unit normal at $x \in \partial^*\Omega$ (see [1]). Notice that, if we consider a rescaled set $\lambda\Omega$ with $\lambda > 0$, we have

$$\mathrm{E}_p(\lambda\Omega) = \lambda^{2n-2}\mathrm{E}_p(\Omega).$$

A natural question is understanding the range of possible values of $\mathrm{E}_p(\Omega)$ for sets Ω with a prescribed perimeter or volume. In particular, we shall prove that there exist optimal constants $c(n,p)$ and $C(n,p)$, depending only on n and p, such that

$$c(n,p)\,\mathrm{P}(\Omega)^2 \le \mathrm{E}_p(\Omega) \le C(n,p)\,\mathrm{P}(\Omega)^2 \tag{2}$$

for every set $\Omega \subset \mathbb{R}^n$ of finite perimeter. For some values of p, we can also characterize the optimal sets realizing the equalities in (2), that is, we consider the variational problems

$$\min\left\{\mathrm{E}_p(\Omega):\ \Omega \subset \mathbb{R}^n,\ |\Omega| < +\infty,\ \mathrm{P}(\Omega) = P_0 \in (0,+\infty)\right\}, \tag{3}$$

$$\max\left\{\mathrm{E}_p(\Omega):\ \Omega \subset \mathbb{R}^n,\ |\Omega| < +\infty,\ \mathrm{P}(\Omega) = P_0 \in (0,+\infty)\right\}. \tag{4}$$

These problems reduce to the study of a quadratic functional on probability measures on the unit sphere $\mathbb{S}^{n-1}$. Indeed, letting $\mu_\Omega \in \mathcal{P}(\mathbb{S}^{n-1})$ be the push-forward of the normalized area measure $\mathrm{P}(\Omega)^{-1}\,\mathcal{H}^{n-1}\lfloor_{\partial^*\Omega}$ via the Gauss map ν_Ω (see Section 2 for the precise definition), we have

$$\mathrm{E}_p(\Omega) = \mathrm{P}(\Omega)^2 J_p(\mu_\Omega), \qquad \text{where} \quad J_p(\mu) := \iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} |v-w|^p\, d\mu(v)d\mu(w),$$

with the constraint $\int_{\mathbb{S}^{n-1}} v\, d\mu_\Omega(v) = 0$. Thus, the constants in (2) are given by

$$c(n,p) = \min_{\mu\in\mathcal{P}_0(\mathbb{S}^{n-1})} J_p(\mu), \qquad C(n,p) = \max_{\mu\in\mathcal{P}_0(\mathbb{S}^{n-1})} J_p(\mu), \tag{5}$$

where $\mathcal{P}_0(\mathbb{S}^{n-1})$ denotes the set of probability measures on $\mathbb{S}^{n-1}$ with barycenter in the origin.

The paper is organized as follows: in Section 2 we set up the notation and recall some basic facts about sets of finite perimeter and Radon measures. In Section 4 we give some estimates and, in some cases, we characterize the constants $c(n,p)$ and $C(n,p)$ for different ranges of p. In Section 5.1 we characterize the optimal sets for problems (3) and (4), and in Section 5.2 we consider the corresponding minimum problem with a volume constraint. Finally, in Section 5.3 we discuss analogous problems for closed curves in $\mathbb{R}^n$ of fixed length.

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2. Notation

Let $n \ge 2$. Given a measurable set $\Omega \subset \mathbb{R}^n$, its perimeter $\mathrm{P}(\Omega)$ is defined as

$$\mathrm{P}(\Omega) = \sup\left\{\int_{\mathbb{R}^n} \chi_\Omega \operatorname{div} g\, dx:\ g \in C^1_c(\mathbb{R}^n;\mathbb{R}^n),\ \|g\|_\infty \le 1\right\}$$

where χ_Ω is the characteristic function of Ω. If $\mathrm{P}(\Omega) < +\infty$, we say that Ω is a set of finite perimeter. In this case, there exists a $\mathcal{H}^{n-1}$-rectifiable set $\partial^*\Omega$, called reduced boundary, and a measurable function $\nu_\Omega : \partial^*\Omega \to \mathbb{S}^{n-1}$, called measure-theoretic outer unit normal. The energy E_p in (1) is well-defined for any set Ω of finite perimeter, since the normal map ν_Ω is defined $\mathcal{H}^{n-1}$-almost everywhere on $\partial^*\Omega$. Gauss-Green Theorem ensures that, if Ω has finite measure, then $\int_{\partial^*\Omega} \nu_\Omega(x)\, d\mathcal{H}^{n-1}(x) = 0$.

We denote by $\mathcal{P}(\mathbb{S}^{n-1})$ the set of probability measures on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, and by $\mathcal{P}_0(\mathbb{S}^{n-1})$ the subset of $\mathcal{P}(\mathbb{S}^{n-1})$ with barycenter in the origin. We recall that, for $\mu \in \mathcal{P}(\mathbb{S}^{n-1})$, the barycenter of μ is the vector

$$\overline{\mu} = \int_{\mathbb{S}^{n-1}} v \, d\mu(v) \in \mathbb{R}^n.$$

Given a set Ω of finite volume and finite perimeter, we define the push-forward of the area measure $\mathcal{H}^{n-1}$ of $\partial^*\Omega$ with respect to its measure-theoretic outer unit normal by

$$\mu_\Omega := \frac{(\nu_\Omega)_\# \mathcal{H}^{n-1} \lfloor_{\partial^*\Omega}}{\mathrm{P}(\Omega)} \in \mathcal{P}_0(\mathbb{S}^{n-1}).$$

Notice that a measure μ_Ω corresponding to a set Ω of finite perimeter cannot be supported on a closed hemisphere.

3. A variational problem on probability measures

For $p > 0$ and $\mu \in \mathcal{P}(\mathbb{S}^{n-1})$, we define

$$J_p(\mu) := \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} |v - w|^p \, d\mu(v) d\mu(w). \tag{6}$$

Since the kernel $|v - w|^p$ is continuous on $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$, the functional J_p is continuous with respect to the weak-star convergence of measures.

Two particular measures will play a special role: the uniform measure σ_{n-1} on $\mathbb{S}^{n-1}$, defined as $\sigma_{n-1} := \mathcal{H}^{n-1}(\mathbb{S}^{n-1})^{-1} \mathcal{H}^{n-1} \lfloor_{\mathbb{S}^{n-1}}$; the measure μ_{sim}, defined up to rotations, uniformly distributed on the vertices of a regular n-simplex inscribed in $\mathbb{S}^{n-1}$. More precisely, letting $v_0, \ldots, v_n \in \mathbb{S}^{n-1}$ be such that $|v_i - v_j|$ is constant for $i \neq j$, then $\mu_{\mathrm{sim}} = \frac{1}{n+1} \sum_{i=0}^n \delta_{v_i}$. Explicit computations yield the values

$$J_p(\sigma_{n-1}) = \frac{2^{p+n-2} \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{p+n-1}{2}\right)}{\sqrt{\pi}\, \Gamma\left(\frac{p}{2} + n - 1\right)},$$

$$J_p(\mu_{\mathrm{sim}}) = 2^{\frac{p}{2}} \left(1 + \frac{1}{n}\right)^{\frac{p}{2} - 1}.$$

The following result, which follows directly from the definition of push-forward measure, shows that the energy $\mathrm{E}_p(\Omega)$ can be expressed in terms of the measure μ_Ω.

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a set of finite perimeter. Then*

$$\mathrm{E}_p(\Omega) = \mathrm{P}(\Omega)^2 \, J_p(\mu_\Omega).$$

The following result provides a characterization of such measures (see [18, Theorems 7.2.1 and 8.2.2]).

Theorem 3.2 (Minkowski Theorem)**.** *Let $\mu \in \mathcal{P}_0(\mathbb{S}^{n-1})$ be such that the support of μ is not contained in a closed hemisphere. Then there exists a bounded convex set $C \subset \mathbb{R}^n$ such that $\mathrm{P}(C) = 1$ and $\mu = \mu_C$, i.e.,*

$$\mu(S) = \mathcal{H}^{n-1}(\{x \in \partial^* C \,:\, \nu_C(x) \in S\})$$

for any Borel subset $S \subset \mathbb{S}^{n-1}$. Moreover the convex set C is unique up to translations.

Remark 3.3. As an application of Theorem 3.2 we obtain that, given $\Omega \subset \mathbb{R}^n$ of finite perimeter, there exists a bounded convex set $C \subset \mathbb{R}^n$, unique up to translations, such that

$$\mu_\Omega = \mu_C, \qquad \mathrm{P}(\Omega) = \mathrm{P}(C), \qquad \mathrm{E}_p(\Omega) = \mathrm{E}_p(C).$$

Indeed, by applying the Minkowski theorem to the measure μ_Ω, we find a bounded convex set $\widehat{C} \subset \mathbb{R}^n$ such that $\mu_\Omega = \mu_{\widehat{C}}$ and $\mathrm{P}(\widehat{C}) = 1$. Setting now $C = \mathrm{P}(\Omega)^{\frac{1}{n-1}}\,\widehat{C}$, we have $\mathrm{P}(\Omega) = \mathrm{P}(C)$ and

$$\mathrm{E}_p(\Omega) = \mathrm{P}(\Omega)^2\, J_p(\mu_\Omega) = \mathrm{P}(C)^2\, J_p(\mu_C) = \mathrm{E}_p(C).$$

Proposition 3.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded set of finite perimeter and let $C \subset \mathbb{R}^n$ be the convex body in Remark 3.2, such that $\mathrm{P}(C) = \mathrm{P}(\Omega)$ and $\mathrm{E}_p(\Omega) = \mathrm{E}_p(C)$. Then, we have*

$$|C| \geq |\Omega|, \tag{7}$$

and equality holds if and only if $\Omega = C$, up to translations and up to a set of measure zero.

Proof. Let $h_C : \mathbb{S}^{n-1} \to \mathbb{R}$ be the support function of the set C, defined as

$$h_C(u) = \sup_{y \in C} y \cdot u.$$

A classical result in convex geometry (see [18, Remark 5.1.2]) expresses the volume of C in terms of its support function and its surface area:

$$\begin{aligned} |C| &= \frac{\mathrm{P}(C)}{n} \int_{\mathbb{S}^{n-1}} h_C(u)\, d\mu_C(u) \\ &= \frac{\mathrm{P}(\Omega)}{n} \int_{\mathbb{S}^{n-1}} h_C(u)\, d\mu_\Omega(u) \\ &= \frac{1}{n} \int_{\partial^*\Omega} h_C(\nu_\Omega(x))\, d\mathcal{H}^{n-1}(x) = \frac{1}{n}\, \mathrm{P}_C(\Omega), \end{aligned} \tag{8}$$

where we denote by $P_C(\Omega)$ the anisotropic perimeter of Ω with respect to the surface tension h_C (with this notation, C corresponds to the Wulff shape of P_C). See [12, Proposition 2.6] for this correspondence. Here we have used that $\mu_\Omega = \mu_C$ and $\mathrm{P}(\Omega) = \mathrm{P}(C)$.

By Wulff's inequality (see for instance [6, 12]), we obtain that

$$\mathrm{P}_C(\Omega) \geq n\, |C|^{\frac{1}{n}}\, |\Omega|^{\frac{n-1}{n}} \tag{9}$$

and the equality holds if and only if Ω is homothetic to C. From (8), we know that $\mathrm{P}_C(\Omega) = n\,|C|$ and thus, (9), we obtain

$$n\,|C| = \mathrm{P}_C(\Omega) \geq n\,|C|^{\frac{1}{n}}\, |\Omega|^{\frac{n-1}{n}},$$

which implies that

$$|C| \geq |\Omega|.$$

Here we have assumed $|C| > 0$; otherwise Ω is negligible and the inequality is trivial.

If $|C| = |\Omega|$, then

$$\mathrm{P}_C(\Omega) = \mathrm{P}_C(C),$$

so that Ω must be equal to C, up to translations and up to negligible sets. □

As shown in [18, Theorem 4.2.1], the weak-star convergence of probability measures is equivalent to the convergence of the corresponding convex bodies.

Theorem 3.5. *Let $\mu \in \mathcal{P}_0(\mathbb{S}^{n-1})$ and let $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{P}_0(\mathbb{S}^{n-1})$. Then, $\mu_n \to \mu$ in the weak-star topology if and only if $\chi_{C_n} \to \chi_C$ in L^1 where C_n (respectively C) are the convex sets corresponding to μ_n (respectively μ) as in Theorem 3.2.*

Definition 3.6. *For $n \geq 2$ and $p > 0$, we define the constants*

$$c(n,p) := \inf_{\mu\in\mathcal{P}_0(\mathbb{S}^{n-1})} J_p(\mu), \qquad C(n,p) := \sup_{\mu\in\mathcal{P}_0(\mathbb{S}^{n-1})} J_p(\mu).$$

By the compactness of $\mathcal{P}_0(\mathbb{S}^{n-1})$ and the continuity of J_p with respect to the weak-star convergence of measures, the infimum and the supremum are actually attained. With this definitions, inequality (2) follows immediately.

Theorem 3.7. *For any set Ω of finite perimeter, we have*

$$c(n,p)\,\mathrm{P}(\Omega)^2 \leq \mathrm{E}_p(\Omega) \leq C(n,p)\,\mathrm{P}(\Omega)^2.$$

Moreover, for any $\varepsilon > 0$ there exist $\Omega_\varepsilon^-, \Omega_\varepsilon^+$ such that

$$\mathrm{E}_p(\Omega_\varepsilon^-) \leq (c(n,p)+\varepsilon)\,\mathrm{P}(\Omega_\varepsilon^-)^2, \qquad \mathrm{E}_p(\Omega_\varepsilon^+) \geq (C(n,p)-\varepsilon)\,\mathrm{P}(\Omega_\varepsilon^+)^2.$$

Remark 3.8. The results in this section still hold for functionals of the form

$$\mathrm{E}_f(\Omega) := \iint_{\partial^*\Omega\times\partial^*\Omega} f(|\nu_\Omega(x)-\nu_\Omega(y)|)\,d\mathcal{H}^{n-1}(x)\,d\mathcal{H}^{n-1}(y),$$

where $f : [0,2] \to \mathbb{R}$ is a given continuous function.

4. Estimates for the optimal values

Proposition 4.1. *For all $p > 0$ we have*

$$\min\{2^{\frac{p}{2}}, 2^{p-1}\} \leq c(n,p) \leq C(n,p) \leq 2^p.$$

Proof. For all $v, w \in \mathbb{S}^{n-1}$ we have $|v-w|^p = 2^{\frac{p}{2}}(1-v\cdot w)^{\frac{p}{2}}$. For any $\mu \in \mathcal{P}_0(\mathbb{S}^{n-1})$, and $p \geq 2$, by Jensen's inequality we have

$$\iint (1-v\cdot w)^{\frac{p}{2}}\,d\mu(v)\,d\mu(w) \geq \left(\iint (1-v\cdot w)\,d\mu(v)\,d\mu(w)\right)^{\frac{p}{2}} = 1^{\frac{p}{2}} = 1,$$

since $\iint v\cdot w\,d\mu(v)\,d\mu(w) = |\overline{\mu}|^2 = 0$. Thus $J_p(\mu) \geq 2^{\frac{p}{2}}$.
For $p < 2$ we will see in Proposition 4.4 below that $c(n,p) = 2^{p-1} < 2^{\frac{p}{2}}$.

The upper bound $C(n,p) \leq 2^p$ is trivial since $|v-w| \leq 2$ for all v, w. □

We now consider different ranges of p.

Proposition 4.2. *For $p = 2$, we have*

$$J_2(\mu) = 2$$

for any $\mu \in \mathcal{P}_0(\mathbb{S}^{n-1})$. Consequently, $c(n,2) = C(n,2) = 2$.

Proof. Compute:

$$|v-w|^2 = 2(1-v\cdot w).$$

Integrating with respect to $\mu\otimes\mu$ gives

$$J_2(\mu) = 2\left(1 - \int v\,d\mu(v)\cdot\int w\,d\mu(w)\right) = 2.$$

□

The following Lemma is taken from [2, Lemma 1] (see also [13, 5]).

Lemma 4.3. *Let $0 < p < 2$ and let ν be a real measure in $\mathbb{S}^{n-1}$ with $\nu(\mathbb{S}^{n-1}) = 0$. Then $J_p(\nu) \le 0$ and the equality holds if and only if $\nu = 0$.*

Proposition 4.4. *For $0 < p < 2$, we have*

$$c(n,p) = 2^{p-1},$$

and the minimum is attained only by measures of the form $\mu = \frac{1}{2}(\delta_v + \delta_{-v})$ for some $v \in \mathbb{S}^{n-1}$. Moreover,

$$C(n,p) = J_p(\sigma_{n-1}).$$

Proof. For any $\mu \in \mathcal{P}_0(\mathbb{S}^{n-1})$, write $|v-w|^p = 2^p \left(\frac{|v-w|}{2}\right)^p$. Consider the inequality

$$t^p \ge 2^{p-2} t^2 \quad \text{for } t \in [0,2],$$

which follows from the fact that the function $t \mapsto t^p/t^2 = t^{p-2}$ is decreasing on $[0,2]$, and equals 2^{p-2} at $t = 2$. Thus

$$|v-w|^p \ge 2^{p-2}|v-w|^2.$$

Integrating both sides of the above inequality with respect to $(v,w) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ gives

$$J_p(\mu) \ge 2^{p-2} J_2(\mu) = 2^{p-2} \cdot 2 = 2^{p-1},$$

since $J_2(\mu) = 2$ for any $\mu \in \mathcal{P}_0(\mathbb{S}^{n-1})$. Equality holds if and only if $|v-w|$ is either 0 or 2 $(\mu \otimes \mu)$-almost everywhere, i.e., μ is supported on two antipodal points. Such a measure has zero barycenter only if the two points are opposite and carry equal mass $\frac{1}{2}$. This proves the statement for $c(n,p)$.

Regarding the maximum, we claim that the uniform measure σ_{n-1} maximizes J_p when $0 < p < 2$. We first show that the energy J_p is concave in $\mathcal{P}_0(\mathbb{S}^{n-1})$. Indeed, let μ_1, $\mu_2 \in \mathcal{P}_0(\mathbb{S}^{n-1})$ and set $\nu := \mu_1 - \mu_2$. By definition, we have that $\nu(\mathbb{S}^{n-1}) = 0$. By Lemma 4.3, we obtain

$$J_p(\nu) \le 0.$$

We then get

$$\begin{aligned} J_p(t\,\mu_1 + (1-t)\,\mu_2) &= t^2\, J_p(\mu_1) + (1-t)^2\, J_p(\mu_2) \\ &\quad + 2\,t\,(1-t) \iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} |v-w|^p\, d\mu_1(v)\, d\mu_2(w) \\ &= t^2\, J_p(\mu_1) + (1-t)^2\, J_p(\mu_2) + t\,(1-t)\, \left(J_p(\mu_1) + J_p(\mu_2) - J_p(\nu)\right) \\ &= t\, J_p(\mu_1) + (1-t)\, J_p(\mu_2) - t\,(1-t)\, J_p(\nu) \\ &\ge t\, J_p(\mu_1) + (1-t)\, J_p(\mu_2) \end{aligned} \tag{10}$$

for any $t \in [0,\, 1]$, which proves the concavity of J_p. Since the set $\mathcal{P}_0(\mathbb{S}^{n-1})$ is convex and J_p is continuous, there exists a unique maximizer which is necessarily the uniform measure σ_{n-1}. □

Remark 4.5. Notice that the measure $\mu = \frac{1}{2}(\delta_v + \delta_{-v})$ cannot be realized as μ_Ω for a set Ω of finite perimeter, because this forces the set to be an unbounded slab, which has infinite perimeter. Nevertheless, the constant $c(n,p)$ is optimal in the sense that there exist sequences of sets Ω_k with $\mathrm{P}(\Omega_k) = 1$ for all k and $\mathrm{E}_p(\Omega_k) \to 2^{p-1}$ as $k \to +\infty$.

For $p > 2$, antipodal measures are the unique maximizers.

Proposition 4.6. *For $p > 2$ we have*

$$C(n,p) = 2^{p-1},$$

and the maximum is attained only by measures of the form $\mu = \frac{1}{2}(\delta_v + \delta_{-v})$ for $v \in \mathbb{S}^{n-1}$.

Proof. The result is contained in [5, Theorem 4.6.6] for the proof; however we repeat the proof here for convenience. We first notice that

$$|v-w|^p \le 2^{p-2}|v-w|^2,$$

which follows from $t^p \le 2^{p-2}t^2$ for $t \in [0,2]$, since t^{p-2} is strictly increasing for $p > 2$. Thus,

$$J_p(\mu) \le 2^{p-2}J_2(\mu) = 2^{p-1},$$

with equality if and only if $|v-w|$ is equal to either 0 or 2 for $(\mu\otimes\mu)$-a.e. $(v,\, w) \in \mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$, i.e. μ is supported on two antipodal points, hence $C(n,p) = 2^{p-1}$. □

For $2 < p < 4$, the uniform measure σ_{n-1} becomes the unique minimizer, while the maximum is still achieved by the antipodal measure.

Proposition 4.7. *For $2 < p < 4$ we have $c(n,p) = J_p(\sigma_{n-1})$, and σ_{n-1} is the unique minimizer.*

Proof. The proof is similar to that of Proposition 4.4. In Lemma 4.8 below we show that, for $2 < p < 4$, we have

$$J_p(\nu) \ge 0 \tag{11}$$

for any signed measure ν with

$$\nu(\mathbb{S}^{n-1}) = 0 \quad \text{and} \quad \int_{\mathbb{S}^{n-1}} v\, d\nu(v) = 0. \tag{12}$$

The equality holds if and only if $\nu = 0$.

Let $\mu_1,\, \mu_2 \in \mathcal{P}_0(\mathbb{S}^{n-1})$ and set $\nu := \mu_1 - \mu_2$. For the same reason as in the proof of Proposition 4.4, we have

$$\begin{aligned} J_p\left(t\,\mu_1 + (1-t)\,\mu_2\right) &= t\,J_p(\mu_1) + (1-t)J_p(\mu_2) - t\,(1-t)\,J_p(\nu) \\ &\le t\,J_p(\mu_1) + (1-t)\,J_p(\mu_2) \end{aligned}$$

for any $t \in [0,\, 1]$. Here we have used (11) in the last inequality. Thus, it follows that J_p is strictly convex on $\mathcal{P}_0(\mathbb{S}^{n-1})$. Since the set $\mathcal{P}_0(\mathbb{S}^{n-1})$ is convex and J_p is continuous, there exists a unique minimizer which is necessarily the uniform measure σ_{n-1}. □

Lemma 4.8. *For $2 < p < 4$, the inequality (11) holds for any signed measure ν on $\mathbb{S}^{n-1}$ satisfying the moment condition (12). Moreover, the equality in (11) holds if and only if $\nu = 0$.*

Proof. The argument is based on the representation of $J_p(\nu)$ by Fourier transform.

We first observe that the polynomial $|v-w|^p$, for $2 < p < 4$, can be expressed as

$$|v-w|^p = K(n,p)\int_{\mathbb{R}^n} \frac{1-\cos\bigl(\xi\cdot(v-w)\bigr) - \frac{1}{2}\bigl(\xi\cdot(v-w)\bigr)^2}{|\xi|^{n+p}}\, d\xi, \tag{13}$$

with a constant $K(n,p) < 0$.

To check this, we first see that

$$1-\cos\bigl(\xi\cdot z\bigr) - \frac{1}{2}\bigl(\xi\cdot z\bigr)^2 = \mathrm{O}(|\xi|^4)$$

for any $\xi \in \mathbb{R}^n$ sufficiently small and any $z \in \mathbb{R}^n$. Thus, we have

$$\int_{B_\varepsilon(0)} \frac{1-\cos\big(\xi\cdot z\big)-\frac{1}{2}\big(\xi\cdot z\big)^2}{|\xi|^{n+p}}\,d\xi \le C' \int_{B_\varepsilon(0)} \frac{d\xi}{|\xi|^{n+p-4}} < +\infty$$

for $\varepsilon > 0$ sufficiently small where C' is a positive constant since $p < 4$. On the other hand, we have

$$\int_{B^c_\varepsilon(0)} \frac{1-\cos\big(\xi\cdot z\big)-\frac{1}{2}\big(\xi\cdot z\big)^2}{|\xi|^{n+p}}\,d\xi \le C'' \int_{B^c_\varepsilon(0)} \frac{d\xi}{|\xi|^{n+p-2}} < +\infty$$

for any $\varepsilon > 0$ where C'' is a positive constant since $p > 2$. Therefore, the right-hand side of (13) is absolutely convergent for any $v,\, w \in \mathbb{S}^{n-1}$ when $2 < p < 4$. Moreover we notice, by using the rotation map $R(|\xi|\,e_1) = \xi$ for any $\xi \in \mathbb{R}^n$ where $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^n$ and the change of variables, that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{1-\cos\big(\xi\cdot(v-w)\big)-\frac{1}{2}\big(\xi\cdot(v-w)\big)^2}{|\xi|^{n+p}}\,d\xi \\
&= \int_{\mathbb{R}^n} \frac{1-\cos(R^{-1}(\eta)\cdot(v-w))-\frac{1}{2}\left(R^{-1}(\eta)\cdot(v-w)\right)^2}{|\eta|^{n+p}}\,d\eta \\
&= \int_{\mathbb{R}^n} \frac{1-\cos(\eta\cdot(|v-w|\,e_1))-\frac{1}{2}\left(\eta\cdot(|v-w|\,e_1)\right)^2}{|\eta|^{n+p}}\,d\eta \\
&= |v-w|^p \int_{\mathbb{R}^n} \frac{1-\cos(\xi\cdot e_1)-\frac{1}{2}\left(\xi\cdot e_1\right)^2}{|\xi|^{n+p}}\,d\xi \\
&=: K(n,p)\,|v-w|^p
\end{aligned}$$

for any $v,\, w \in \mathbb{S}^{n-1}$. Now we compute the constant $C(n,p)$ and show that $C(n,p)$ is negative. By the Fubini-Tonelli theorem, we have

$$\begin{aligned}
K(n,p) &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \frac{1-\cos(\xi_1)-\frac{1}{2}\xi_1^2}{|\xi_1|^{n+p}\left(1+\frac{|\xi'|^2}{\xi_1^2}\right)^{\frac{n+p}{2}}}\,d\xi_1\,d\xi' \\
&= \int_{\mathbb{R}^{n-1}} \frac{d\xi'}{(1+|\xi'|^2)^{\frac{n+p}{2}}} \int_{\mathbb{R}} \frac{1-\cos x-\frac{1}{2}x^2}{|x|^{1+p}}\,dx.
\end{aligned}$$

Since $2 < p < 4$, by using the integration-by-part formula, we obtain

$$\int_{\mathbb{R}} \frac{1-\cos x-\frac{1}{2}x^2}{|x|^{1+p}}\,dx = \frac{2}{p\,(p-1)} \int_0^\infty \frac{\cos x - 1}{x^{p-1}}\,dx = \frac{-2}{p\,(p-1)} \int_0^\infty \frac{1-\cos x}{x^{p-1}}\,dx < 0,$$

and thus $K(n,p) < 0$.

For a signed measure ν on $\mathbb{S}^{n-1}$, we define its Fourier transform $\widehat{\nu}$ as

$$\widehat{\nu}(\xi) = \int_{\mathbb{S}^{n-1}} e^{-2\pi\,i\,\xi\cdot v}\,d\nu(v)$$

for $\xi \in \mathbb{R}^n$ (see, for instance, [20, Chapter II]). Since ν is compactly supported, $\widehat{\nu}$ is an entire function of exponential type; in particular, $\widehat{\nu}$ is of class C^∞ on $\mathbb{R}^n$ and all its derivatives are bounded.

Now, inserting (13) into the definition of $J_p(\nu)$ and using the Fubini-Tonelli theorem, we get (14)

$$J_p(\nu) = \iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} \left(K(n,p) \int_{\mathbb{R}^n} \frac{1-\cos\big(\xi\cdot(v-w)\big) - \frac{1}{2}\big(\xi\cdot(v-w)\big)^2}{|\xi|^{n+p}}\, d\xi \right) d\nu(v)\, d\nu(w)$$
$$= \int_{\mathbb{R}^n} \frac{K(n,p)}{|\xi|^{n+p}} \left[\iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} \big(1-\cos(\xi\cdot(v-w)) - \tfrac{1}{2}(\xi\cdot(v-w))^2\big)\, d\nu(v)\, d\nu(w) \right] d\xi$$
$$= \frac{1}{(2\pi)^p} \int_{\mathbb{R}^n} \frac{K(n,p)}{|\xi|^{n+p}} \iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} \left(1-\cos(2\pi\,\xi\cdot(v-w)) - \frac{1}{2}(2\pi\,\xi\cdot(v-w))^2\right) d\nu(v)\, d\nu(w) d\xi.$$

Recalling condition (12) we get

$$\iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} \cos\big(2\pi\,\xi\cdot(v-w)\big) d\nu(v) d\nu(w) = \Re\Big(\iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} e^{2\pi\, i\,\xi\cdot(v-w)} d\nu(v) d\nu(w) \Big)$$
$$= \Re\big(\widehat{\nu}(\xi)\,\overline{\widehat{\nu}(\xi)}\big) = |\widehat{\nu}(\xi)|^2,$$

where $\Re(z)$ denotes the real part of z, and

$$\iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} (2\pi\xi\cdot(v-w))^2\, d\nu(v)\, d\nu(w) = 2\int_{\mathbb{S}^{n-1}} (2\pi\xi\cdot v)^2\, d\nu(v) \int_{\mathbb{S}^{n-1}} d\nu(w)$$
$$- 2\left(\int_{\mathbb{S}^{n-1}} 2\pi\xi\cdot v\, d\nu(v) \right)^2 = 0.$$

Plugging these identities into (14), we eventually obtain

$$J_p(\nu) = -\frac{K(n,p)}{(2\pi)^p} \int_{\mathbb{R}^n} \frac{|\widehat{\nu}(\xi)|^2}{|\xi|^{n+p}}\, d\xi. \tag{15}$$

Since $K(n,p) < 0$ and the integrand is non-negative, we conclude that $J_p(\nu) \geq 0$.

If $J_p(\nu) = 0$, then $|\widehat{\nu}(\xi)|^2 = 0$ for a.e. $\xi \in \mathbb{R}^n$, so that $\widehat{\nu} = 0$, namely, $\nu = 0$. □

Proposition 4.9. *For $p = 4$, we have*

$$c(n,4) = 4 + \frac{4}{n},$$

and a measure $\mu \in \mathcal{P}_0(\mathbb{S}^{n-1})$ minimizes J_4 if and only if it satisfies the isotropy condition

$$\iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} v_i\, w_j\, d\mu(v)\, d\mu(w) = \frac{1}{n}\delta_{ij}, \qquad i,j = 1,\ldots,n. \tag{16}$$

In particular, both σ_{n-1} and μ_{sim} are minimizers.

Proof. We compute

$$|v-w|^4 = (2-2v\cdot w)^2 = 4(1-v\cdot w)^2 = 4(1-2v\cdot w + (v\cdot w)^2).$$

Integrating and using $\int_{\mathbb{S}^{n-1}} v\, d\mu(v) = 0$ gives

$$J_4(\mu) = 4\left(1 + \iint (v\cdot w)^2\, d\mu(v)\, d\mu(w)\right).$$

Let now M be the matrix with entries $M_{ij} = \int v_i v_j\, d\mu(v)$. Then

$$\iint (v\cdot w)^2\, d\mu(v) d\mu(w) = \sum_{i,j} (M_{ij})^2_{1\leq i,\, j\leq n} = \mathrm{tr}(M^2).$$

Note that M is symmetric, positive semidefinite, and $\mathrm{tr}(M) = \int_{\mathbb{S}^{n-1}} |v|^2 \, d\mu(v) = 1$. By the Cauchy–Schwarz inequality, we have

$$\mathrm{tr}(M^2) \geq \frac{1}{n}(\mathrm{tr}(M))^2 = \frac{1}{n}, \tag{17}$$

with equality if and only if $M = \frac{1}{n}I$. Thus

$$J_4(\mu) = 4\left(1 + \mathrm{tr}(M^2)\right) \geq 4\left(1 + \frac{1}{n}\right),$$

with equality if and only if $M = \frac{1}{n}I$, i.e.,

$$\int_{\mathbb{S}^{n-1}} v_i \, v_j \, d\mu(v) = \frac{1}{n}\delta_{ij}.$$

One can check that, due to symmetry, for σ_{n-1} and μ_{sim} it holds $M = \frac{1}{n}I$, thus both measures are minimizers. □

For $p > 4$, the minimization problem becomes more delicate. We first state the following auxiliary result.

Lemma 4.10. *Let $p > 4$, $A \in (0, 1]$ and $f(t) := (1-t)^{\frac{p}{2}}$. Then every minimizer ν of*

$$F(A) := \min\left\{\int_{-1}^{1} f(t)\, d\nu(t) : \nu \in \mathcal{P}([-1,1]), \int t \, d\nu = 0, \int t^2 \, d\nu = A\right\}$$

is supported on exactly two points.

Proof. Recall that the extreme points of $\mathcal{P}([-1, 1])$ are Dirac measures on $[-1, 1]$ (see for instance [11, Proposition 10.1.3]), and define

$$\mathcal{A} := \left\{\nu \in \mathcal{P}([-1,1]) : \int t \, d\nu = 0, \int t^2 \, d\nu = A\right\}.$$

Since $\mathcal{A}$ is the intersection of $\mathcal{P}([-1,1])$ with two closed hyperplanes, by [10, Main Theorem] (see also [21]) we have that any extreme point of $\mathcal{A}$ is supported on at most three extreme points of $\mathcal{P}([-1,1])$, that is, it is supported in at most three distinct points. Moreover, by linearity of the energy in $F(A)$, the minimum is attained on the extreme points of $\mathcal{A}$.

We now show that a minimizer of the energy that is also an extreme point of $\mathcal{A}$ has exactly two distinct points in its support. First of all, it is easy to see that such minimizer cannot be a measure supported only on one point because of the constraint. We now suppose, by contradiction, that a minimizer ν is supported on three distinct points $t_1 < t_2 < t_3$ with weights $p_1, p_2, p_3 > 0$ such that $p_1 + p_2 + p_3 = 1$. Consider the function

$$\mathcal{L}(\nu, \lambda_0, \lambda_1, \lambda_2) = \int_{-1}^{1} f(t)\, d\nu(t) - \lambda_0 \left(\int_{-1}^{1} d\nu(t) - 1\right) - \lambda_1 \int_{-1}^{1} t \, d\nu(t) - \lambda_2 \left(\int_{-1}^{1} t^2 \, d\nu(t) - A\right),$$

where $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{R}$ are Lagrange multipliers. For the measure $\nu = \sum_{i=1}^{3} p_i \delta_{t_i}$, $\mathcal{L}$ becomes a function of the parameters $\{p_i, t_i\}$

$$L(p_1, p_2, p_3, t_1, t_2, t_3) = \sum_{i=1}^{3} p_i f(t_i) - \lambda_0 \left(\sum_{i=1}^{3} p_i - 1\right) - \lambda_1 \sum_{i=1}^{3} p_i t_i - \lambda_2 \left(\sum_{i=1}^{3} p_i t_i^2 - A\right).$$

At an extreme point, the gradient of L with respect to all free variables must vanish. This yields

$$\frac{\partial L}{\partial p_i} = f(t_i) - \lambda_0 - \lambda_1 t_i - \lambda_2 t_i^2 = 0, \qquad i = 1, 2, 3.$$

Thus for each support point t_i we have

$$f(t_i) = \lambda_0 + \lambda_1 t_i + \lambda_2 t_i^2. \tag{18}$$

Since $t_2 \in (-1, 1)$, we also have

$$\frac{\partial L}{\partial t_2} = p_2 f'(t_2) - \lambda_1 p_2 - 2\lambda_2 p_2 t_2 = 0.$$

Since $p_2 > 0$, we obtain

$$f'(t_2) = \lambda_1 + 2\lambda_2 t_2. \tag{19}$$

Define the quadratic polynomial

$$Q(t) := \lambda_0 + \lambda_1 t + \lambda_2 t^2.$$

Equation (18) implies that $f(t_i) = Q(t_i)$ for $i = 1, 2, 3$, and Equation (19) implies that $f'(t_2) = Q'(t_2)$. Consequently, the difference function

$$g(t) := f(t) - Q(t)$$

satisfies

$$g(t_i) = 0 \quad \text{for } i = 1, 2, 3 \quad \text{and} \quad g'(t_2) = 0.$$

Since g is smooth on $[-1, 1]$, Rolle's Theorem applied to g on the intervals $[t_1, t_2]$ and $[t_2, t_3]$ yields points $\eta_1 \in (t_1, t_2)$ and $\eta_2 \in (t_2, t_3)$ such that

$$g'(\eta_1) = g'(\eta_2) = 0.$$

Applying again Rolle's Theorem to the function g' on the intervals $[\eta_1, t_2]$ and $[t_2, \eta_2]$, we find points $\xi_1 \in (\eta_1, t_2)$ and $\xi_2 \in (t_2, \eta_2)$ such that

$$g''(\xi_1) = g''(\xi_2) = 0.$$

However, we have

$$g''(t) = f''(t) - 2\lambda_2.$$

and

$$f''(t) = \frac{p}{2}\left(\frac{p}{2} - 1\right)(1 - t)^{\frac{p}{2}-2}$$

for any $t \in [-1, 1]$. Since $\frac{p}{2} - 2 > 0$, the factor $(1 - t)^{\frac{p}{2}-2}$ is strictly decreasing on $[-1, 1]$. Moreover, $\frac{p}{2}(\frac{p}{2} - 1) > 0$, hence $f''(t)$ itself is strictly decreasing on $[-1, 1]$. Consequently, the equation $g''(t) = f''(t) - 2\lambda_2 = 0$ can have at most one solution, contradicting the existence of two distinct points $\xi_1 \neq \xi_2$ with $g''(\xi_1) = g''(\xi_2) = 0$. Therefore, a minimizer cannot be supported in three distinct points.

A completely analogous argument, by restricting to three points contained in the support, excludes that a minimizer is a discrete measure supported in more than two points.

Assume now that there exists a minimizer ν^* which is not an extreme point of $\mathcal{A}$. Then, by Choquet Theorem (see for instance [11, Theorem 10.1.7]) we can write

$$\nu^* = \int_{\mathcal{E}} e \, d\mu(e),$$

where μ is a probability measure on the set $\mathcal{E}$ of the extreme points of $\mathcal{A}$. If ν^* is not an extreme point, then the support of μ contains at least two elements ν_1, ν_2, which are also minimizers (by linearity of the energy). As a consequence, the discrete measure $\frac{\nu_1+\nu_2}{2}$ would be a minimizer supported on three or four points, leading to a contradiction. It follows that every minimizer is an extreme point of $\mathcal{A}$, and it is supported on exactly two points. □

Theorem 4.11. *For $p > 4$ the minimum of J_p is attained uniquely by the measure μ_{sim}. In particular,*

$$c(n,p) = 2^{\frac{p}{2}} \left(1 + \frac{1}{n}\right)^{\frac{p}{2}-1}.$$

Proof. For $\mu \in \mathcal{P}_0(\mathbb{S}^{n-1})$ consider the push-forward measure ν of $\mu \otimes \mu$ under the map $(v,w) \mapsto v \cdot w$. Then ν is a probability measure on $[-1,1]$ satisfying

$$\int_{-1}^{1} t \, d\nu(t) = 0, \qquad \int_{-1}^{1} t^2 \, d\nu(t) = A,$$

where

$$A := \iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} (v\cdot w)^2 \, d\mu(v)\, d\mu(w) \in (0,1].$$

Since $|v-w|^p = (2-2v\cdot w)^{\frac{p}{2}} = 2^{\frac{p}{2}}(1-v\cdot w)^{\frac{p}{2}}$, we have

$$J_p(\mu) = 2^{\frac{p}{2}} \int_{-1}^{1} (1-t)^{\frac{p}{2}} \, d\nu(t). \tag{20}$$

For $A \in (0,1]$ we consider the minimum problem

$$F(A) := \min\left\{ \int_{-1}^{1} f(t)\, d\nu(t) \;:\; \nu \in \mathcal{P}([-1,1]),\ \int t\, d\nu = 0,\ \int t^2\, d\nu = A \right\},$$

where $f(t) := (1-t)^{\frac{p}{2}}$. By Lemma 4.10 we can restrict to measures of the form $\nu = p\delta_a + (1-p)\delta_b$ with $a, b \in [-1,1]$, $\lambda \in (0,1)$. The moment conditions give

$$\lambda\, a + (1-\lambda)\, b = 0 \quad \text{and} \quad \lambda\, a^2 + (1-\lambda)\, b^2 = A. \tag{21}$$

From the condition $\lambda\, a + (1-\lambda)\, b = 0$, we get $b = -\dfrac{\lambda}{1-\lambda}\, a$, hence

$$a^2 = \frac{A\,(1-\lambda)}{\lambda}, \quad b = -\frac{A}{a}, \quad \text{and} \quad \lambda = \frac{A}{A+a^2}.$$

By symmetry we can assume $a \geq 0$, then the constraints $a \leq 1$, $b \geq -1$ imply $a \in [A,1]$. Define

$$G(a) := \frac{A}{A+a^2}\, f(a) + \frac{a^2}{A+a^2}\, f\left(-\frac{A}{a}\right).$$

We claim that $\min_{a\in[A,1]} G(a) = G(1)$. Indeed, we shall prove that $G'(a) < 0$ for every $a \in [A,1)$, hence G is strictly decreasing on $[A,1]$. Set $\alpha = \frac{p}{2} > 2$. Then

$$f(t) = (1-t)^{\alpha} \quad \text{and} \quad f'(t) = -\alpha\,(1-t)^{\alpha-1}.$$

Define $D = A + a^2$. The derivative is

$$G'(a) = \frac{2aA}{D^2}\left[f\left(\frac{-A}{a}\right) - f(a)\right] + \frac{A}{D}\left[f'(a) + f'\left(\frac{-A}{a}\right)\right].$$

Since $A > 0$ and $D > 0$, the sign of $G'(a)$ equals the sign of

$$K(a) := \frac{2a}{D}\left[f\left(\frac{-A}{a}\right) - f(a)\right] + f'(a) + f'\left(\frac{-A}{a}\right).$$

Introduce the variables

$$x = 1 - a \quad \text{and} \quad y = 1 + \frac{A}{a}.$$

Then $0 < x \leq 1 - A$ and $y \geq 1 + A$. Moreover,

$$a = 1 - x, \quad \frac{A}{a} = y - 1, \quad \text{and} \quad A = a(y-1) = (1-x)(y-1),$$

and thus we have

$$D = A + a^2 = (1-x)(y-1) + (1-x)^2 = (1-x)(y-x).$$

Hence

$$\frac{2a}{D} = \frac{2(1-x)}{(1-x)(y-x)} = \frac{2}{y-x}.$$

We also have

$$f\left(-\frac{A}{a}\right) = \left(1 + \frac{A}{a}\right)^{\alpha} = y^{\alpha}, \qquad f(a) = (1-a)^{\alpha} = x^{\alpha},$$

and

$$f'\left(\frac{-A}{a}\right) = -\alpha\left(1 + \frac{A}{a}\right)^{\alpha-1} = -\alpha y^{\alpha-1}, \qquad f'(a) = -\alpha\,(1-a)^{\alpha-1} = -\alpha x^{\alpha-1}.$$

Therefore,

$$K(a) = \frac{2}{y-x}(y^{\alpha} - x^{\alpha}) - \alpha\big(x^{\alpha-1} + y^{\alpha-1}\big).$$

Multiplying by the positive quantity $y - x$ gives

$$(y-x)K(a) = 2(y^{\alpha} - x^{\alpha}) - \alpha(y-x)\big(x^{\alpha-1} + y^{\alpha-1}\big).$$

Thus $K(a) < 0$ if and only if

$$2(y^{\alpha} - x^{\alpha}) < \alpha(y-x)\big(x^{\alpha-1} + y^{\alpha-1}\big). \tag{22}$$

Set $t = \frac{x}{y}$ $(0 < t < 1)$. Then $y - x = y(1-t)$ and after division by y^{α}, the inequality (22) becomes

$$2(1 - t^{\alpha}) < \alpha(1-t)\big(t^{\alpha-1} + 1\big). \tag{23}$$

Define $\psi(t) = \alpha(1-t)\big(1 + t^{\alpha-1}\big) - 2(1-t^{\alpha})$ for $t \in (0,1]$. We shall prove $\psi(t) > 0$ for all $t \in (0,1)$. Since $\psi(1) = 0$, it suffices to show that ψ is strictly decreasing on $(0,1)$. Compute

$$\psi'(t) = \alpha\Big[-\big(1 + t^{\alpha-1}\big) + (1-t)(\alpha-1)t^{\alpha-2}\Big] + 2\alpha t^{\alpha-1} = \alpha\Big[-1 + t^{\alpha-1} + (\alpha-1)(1-t)t^{\alpha-2}\Big].$$

We rewrite the bracket as

$$-1 + t^{\alpha-2}\big[t + (\alpha-1)(1-t)\big] = -1 + t^{\alpha-2}\big[(\alpha-1) - (\alpha-2)t\big].$$

Set $g(t) = t^{\alpha-2}\big((\alpha-1) - (\alpha-2)t\big)$. Then $\psi'(t) = \alpha\big(-1 + g(t)\big)$. The derivative of g is

$$\begin{aligned} g'(t) &= (\alpha-2)t^{\alpha-3}\big((\alpha-1) - (\alpha-2)t\big) + t^{\alpha-2}\big(-(\alpha-2)\big) \\ &= (\alpha-2)t^{\alpha-3}\big[(\alpha-1) - (\alpha-2)t - t\big] \\ &= (\alpha-2)(\alpha-1)t^{\alpha-3}(1-t). \end{aligned}$$

For $\alpha > 2$ we have $(\alpha-2)(\alpha-1) > 0$, and for $t \in (0,1)$ also $t^{\alpha-3} > 0$ and $1-t > 0$; hence $g'(t) > 0$. Thus g is strictly increasing on $(0,1]$. Since $g(1) = 1$, it follows that $g(t) < 1$ for every $t \in (0,1)$. Consequently $\psi'(t) = \alpha(-1+g(t)) < 0$ for all $t \in (0,1)$, so ψ is strictly decreasing. Because $\psi(1) = 0$, we obtain $\psi(t) > 0$ for every $t \in (0,1)$, which is exactly inequality (23). Therefore (22) holds, hence $K(a) < 0$ and finally $G'(a) < 0$ for every $a \in [A,1)$. Thus G is strictly decreasing on $[A,1]$, and its minimum is attained at the right endpoint $a = 1$. It follows that

$$F(A) = \min_{a\in[A,1]} G(a) = G(1) = (1+A)^{\alpha-1},$$

and the minimizer corresponds to $a = 1$, $b = -A$, and $p = \frac{A}{1+A}$, i.e.,

$$\nu_A = \frac{A}{1+A}\,\delta_1 + \frac{1}{1+A}\,\delta_{-A}. \tag{24}$$

From (20) and the definition of $F(A)$, we get

$$J_p(\mu) = 2^{\frac{p}{2}} \int_{-1}^{1} f(t)\,d\nu(t) \geq 2^{\frac{p}{2}}\,F(A) = 2^{\frac{p}{2}}(1+A)^{\frac{p}{2}-1}, \tag{25}$$

for all measures $\mu \in \mathcal{P}_0(\mathbb{S}^{n-1})$.

Letting now M be such that $M_{ij} = \int_{\mathbb{S}^{n-1}} v_i v_j\,d\mu(v)$, by (17) we have

$$A = \mathrm{tr}(M^2) \geq \frac{1}{n},$$

with the equality if and only if $M = \frac{1}{n} I_n$ (i.e. μ is isotropic). Since $A \mapsto (1+A)^{\frac{p}{2}-1}$ is increasing for $p > 4$, from (25) we then obtain

$$J_p(\mu) \geq 2^{\frac{p}{2}}\left(1+\frac{1}{n}\right)^{\frac{p}{2}-1}. \tag{26}$$

The equality forces $A = \frac{1}{n}$ and $\nu = \nu_{\frac{1}{n}} = \frac{1}{n+1}\,\delta_1 + \frac{n}{n+1}\,\delta_{-\frac{1}{n}}$. Thus for $(\mu\otimes\mu)$-almost every $(v,w) \in \mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$, $v\cdot w \in \{1, -\frac{1}{n}\}$. Hence the support of μ consists of unit vectors with pairwise inner products $-\frac{1}{n}$, which are the vertices of a regular n-simplex. The barycenter condition $\int_{\mathbb{S}^{n-1}} v\,d\mu(v) = 0$ forces μ to be uniform on these $n+1$ vertices, i.e., $\mu = \mu_{\mathrm{sim}}$.

Any other minimizer μ induces the same measure ν, hence for $\mu\otimes\mu$-almost every (v,w) we have $v\cdot w \in \{1, -\frac{1}{n}\}$, so that

$$|v-w|^2 = 2 - 2v\cdot w \in \left\{0,\ 2\left(1+\frac{1}{n}\right)\right\}.$$

Thus, distinct points in the support of μ are at constant distance $\sqrt{2(1+\frac{1}{n})}$; equivalently, they form a set of unit vectors with pairwise inner product $-\frac{1}{n}$. Such a configuration is, up to rotation, exactly the set of vertices of a regular n-simplex. Moreover, the barycenter constraint forces the measure to be uniform on those vertices, so that the minimum is uniquely attained by μ_{sim}. □

5. Optimal sets for E_p

Building on the results of the previous section, we now characterize minimizers and maximizers for the energy E_p under perimeter or volume constraint.

5.1. **Perimeter constraint.** Given $P_0 > 0$, we consider the problems

$$\min\{\mathrm{E}_p(\Omega) : \Omega \subset \mathbb{R}^n,\ \mathrm{P}(\Omega) = P_0\}, \tag{27}$$

$$\max\{\mathrm{E}_p(\Omega) : \Omega \subset \mathbb{R}^n,\ \mathrm{P}(\Omega) = P_0\}. \tag{28}$$

We partially characterize the optimal sets for different ranges of p.

Theorem 5.1. *For $0 < p < 2$:*

(1) *The infimum of* (27) *is not attained. A minimizing sequence is given by thin cylinders with the heights tending to zero.*
(2) *The ball of perimeter P_0 is a maximizer of* (28)*, and it is the unique maximizer among convex sets.*

Proof. We first prove (1). From Proposition 4.4, we have

$$\inf\{\mathrm{E}_p(\Omega) : \Omega \subset \mathbb{R}^n,\ |\Omega| < +\infty,\ \mathrm{P}(\Omega) = P_0\} \geq P_0^2\, c(n,p) = P_0^2\, 2^{p-1},$$

and $J_p(\mu) = c(n,p)$ is achieved only by measures of the form $\mu = \frac{1}{2}(\delta_v + \delta_{-v})$ for some $v \in \mathbb{S}^{n-1}$. Such measures are not induced by a set of finite perimeter, so the the infimum is not attained. To see this, we construct a minimizing sequence such whose energy converges to $P_0^2\, 2^{p-1}$. We set $\Omega_\varepsilon := [0,\, \varepsilon] \times [0,\, L_\varepsilon]^{n-1}$ where $L_\varepsilon > 0$ satisfying $\lim_{\varepsilon \downarrow 0} L_\varepsilon^{n-1} = \frac{P_0}{2}$, so that $\mathrm{P}(\Omega_\varepsilon) = P_0$. Then, an easy computation gives

$$\begin{aligned} \mathrm{E}_p(\Omega_\varepsilon) &= 2 \iint_{F_\varepsilon^1 \times F_\varepsilon^2} 2^{\frac{p}{2}}\, (1 - \nu_{\Omega_\varepsilon}(x) \cdot \nu_{\Omega_\varepsilon}(y))^{\frac{p}{2}}\, d\mathcal{H}^{n-1}\, d\mathcal{H}^{n-1} + \mathrm{O}(\varepsilon) \\ &= 2^{p+1}\, \mathcal{H}^{n-1}(F_\varepsilon^1)\, \mathcal{H}^{n-1}(F_\varepsilon^2) + \mathrm{O}(\varepsilon), \end{aligned}$$

where $F_\varepsilon^1 := \{0\} \times [0,\, L_\varepsilon]^{n-1}$ and $F_\varepsilon^2 := \{\varepsilon\} \times [0,\, L_\varepsilon]^{n-1}$. Letting $\varepsilon \downarrow 0$, we have

$$\lim_{\varepsilon \downarrow 0} \mathrm{E}_p(\Omega_\varepsilon) = 2^{p-1}\, P_0^2.$$

This completes the proof of the first claim.

Regarding (2), by Proposition 4.4, the maximum of $J_p(\mu)$ is uniquely attained at $J_p(\sigma_{n-1})$, which corresponds to a ball of perimeter P_0. □

Remark 5.2. By Proposition 4.2 we have $\mathrm{E}_2(\Omega) = 2\mathrm{P}(\Omega)^2$, thus every set with perimeter P_0 has the same energy.

Theorem 5.3. *For $2 < p < 4$:*

(1) *The ball of perimeter P_0 is a minimizer of* (27)*, and it is the unique minimizer among convex sets.*
(2) *The supremum in* (28) *is not attained. A maximizing sequence is given by thin cylinders with the heights tending to zero.*

Proof. By Proposition 4.7 the unique minimizer of J_p is σ_{n-1}, and the maximizers are measures of the form $\mu = \frac{1}{2}(\delta_v + \delta_{-v})$ for some $v \in \mathbb{S}^{n-1}$. The thesis then follows as in the proof of Theorem 5.1. □

Theorem 5.4. *For $p \geq 4$:*

(1) *For $p = 4$, both the ball and the regular simplex of perimeter P_0 are minimizers of* (27)*.*
(2) *For $p > 4$, the regular simplex of perimeter P_0 is a minimizer of* (27)*, and it is the unique minimizer among convex sets.*

(3) *For $p \geq 4$, the supremum of* (28) *is not attained. A maximizing sequence is given by thin cylinders with the heights tending to zero.*

Proof. By Proposition 4.9, any measure satisfying the isotropy condition (16) minimizes J_4. The ball induces σ_{n-1} and the simplex induces μ_{sim}, which are both isotropic measures.

By Theorem 4.11, the minimum of J_p is attained uniquely by μ_{sim}, which is induced by the regular simplex.

By Propositions 4.9 and 4.6, the maximum of J_p is attained by measures of the form $\mu = \frac{1}{2}(\delta_v + \delta_{-v})$ for some $v \in \mathbb{S}^{n-1}$, and the thesis follows as in the proof of Theorem 5.1. □

5.2. **Volume Constraint.** Given $V_0 > 0$, we now consider the minimum problem

$$\min\{\mathrm{E}_p(\Omega) : \ \Omega \subset \mathbb{R}^n, \, |\Omega| = V_0\}. \tag{29}$$

Theorem 5.5. *For every $p > 0$ problem* (29) *admits a minimizer, which is bounded and convex.*

Proof. Let $\{\Omega_k\}$ be a minimizing sequence for problem (29). From Remark 3.3 and Proposition 3.4), we can assume that Ω_k are convex sets. Indeed, if C_k are bounded convex sets such that $\mathrm{P}(C_k) = \mathrm{P}(\Omega_k)$, $|C_k| \geq |\Omega_k| = V_0$ and $\mathrm{E}_p(C_k) = \mathrm{E}_p(\Omega_k)$, letting $\widetilde{\Omega}_k = (V_0/|C_k|)^{\frac{1}{n}} C_k$ we have $\mathrm{E}_p(\widetilde{\Omega}_k) \leq \mathrm{E}_p(\Omega_k)$, with equality if and only if Ω_k is convex.

Letting B^{V_0} be the ball of volume V_0, by Proposition 4.1 we have

$$\mathrm{P}(\Omega_k)^2 \leq \frac{\mathrm{E}_p(\Omega_k)}{c(n,p)} \leq \frac{\mathrm{E}_p(B^{V_0})}{\min(2^{\frac{p}{2}}, 2^{p-1})},$$

that is, the perimeter of the sets Ω_k is uniformly bounded. As a consequence, the diameter of Ω_k is also uniformly bounded (see for instance [14]), so that, up to a subsequence, $\chi_{\Omega_k} \to \chi_\Omega$ as $k \to +\infty$ in L^1, where Ω is a bounded convex set of volume V_0. Since there holds $\mathrm{P}(\Omega_k) \to \mathrm{P}(\Omega)$, by Theorem 3.5 it follows that Ω is a solution of problem (29). □

Since $J_2(\mu) = 2$ and $\mathrm{E}_2(\Omega) = 2\mathrm{P}(\Omega)^2$, for $p = 2$ the minimization of E_p reduces to minimizing perimeter at fixed volume, which gives the ball B^{V_0} as unique minimizer. More generally, we can show that the ball is the unique minimizer also for $2 < p \leq 4$.

Theorem 5.6. *For $2 \leq p \leq 4$ the ball is the unique minimizer of* E_p.

Proof. By Propositions 4.7 and 4.9, in this range of p the measure σ_{n-1} is a minimizer of J_p, hence

$$\mathrm{E}_p(\Omega) \geq \mathrm{P}(\Omega)^2 J_p(\sigma_{n-1}) \geq \mathrm{P}(B^{V_0})^2 J_p(\sigma_{n-1}) = \mathrm{E}_p(B^{V_0}),$$

for every finite perimeter set Ω with volume V_0, and the equality holds if and only if Ω is a ball of volume V_0. □

Remark 5.7. In the case $0 < p < 2$ the ball minimizes the perimeter but the antipodal measure minimizes J_p (by Proposition 4.4), so we expect that the minimizer is a ball for p close enough to 2, while it degenerates to a hyperplane of multiplicity 2 as $p \to 0$.

In the case $p > 4$ the ball minimizes the perimeter but the uniform measure on the regular simplex minimizes J_p (by Theorem 4.11), so we expect that the minimizer is a ball for p close enough to 4, and a regular simplex for p large enough.

5.3. **Closed curves of fixed length.** In this section, we consider the minimization and maximization of a geometric oscillation energy for closed rectifiable curves in $\mathbb{R}^n$. We set $\mathcal{A}([0,L];\,\mathbb{R}^n)$ be the collection of rectifiable closed curves of length $L>0$, parametrized by arc-length. For $\gamma\in\mathcal{A}([0,L];\mathbb{R}^n)$, we denote the unit tangent vector of γ at $\gamma(s)$ by $\tau(s)\in\mathbb{S}^{n-1}$, and we define

$$\mathrm{E}_p(\gamma) := \iint_{[0,L]^2} |\tau(s)-\tau(t)|^p\,ds\,dt$$

for $p>0$. We consider the problems

$$\min\{\mathrm{E}_p(\gamma) : \gamma\in\mathcal{A}([0,L];\mathbb{R}^n)\} \tag{30}$$

and

$$\max\{\mathrm{E}_p(\gamma) : \gamma\in\mathcal{A}([0,L];\mathbb{R}^n)\}. \tag{31}$$

As in the case of hypersurfaces, we can relate this problem to a variational problem on probability measures on the sphere. Define the push-forward measure

$$\mu_\gamma = \frac{1}{L}\,\tau_\#(\mathcal{H}^1\lfloor_{[0,L]}) \in \mathcal{P}_0(S^{n-1}),$$

and we have

$$\mathrm{E}_p(\gamma) = L^2 J_p(\mu_\gamma).$$

Conversely, given any $\mu\in\mathcal{P}_0(\mathbb{S}^{n-1})$, there exists a rectifiable curve γ of length L such that $\mu_\gamma=\mu$. Indeed, by the Isomorphism theorem for measures (see [15, Theorem 17.41]), there exists a measurable map $u:[0,L]\to\mathbb{S}^{n-1}$ such that $\mu=\frac{1}{L}u_\#\mathcal{L}^1\lfloor_{[0,L]}$ and $\int_0^L u(s)\,ds=0$. Then one can define $\gamma(s)=\int_0^s u(t)\,dt$, which yields a closed rectifiable curve with tangent field u. Therefore, problems (30) and (31) are equivalent to

$$\min_{\mu\in\mathcal{P}_0(\mathbb{S}^{n-1})} L^2 J_p(\mu) \quad\text{and}\quad \max_{\mu\in\mathcal{P}_0(\mathbb{S}^{n-1})} L^2 J_p(\mu),$$

respectively.

Theorem 5.8 (Minimizing curves)**.** *Let $L>0$ be fixed. The minimum value of E_p among closed rectifiable curves of length L is $L^2c(n,p)$, where $c(n,p)$ is given by Theorem 4.4, Proposition 4.2, Theorem 4.7, Theorem 4.9, and Theorem 4.11 for the respective ranges of p. In particular,*

1. *For $0<p<2$, the minimum is attained by curves that are multiply covered segments of length L (where the length is counted with multiplicity). The minimum value is $2^{p-1}L^2$.*
2. *For $p=2$, every curve of length L has energy $2L^2$ and is a minimizer.*
3. *For $2<p<4$ minimizers are curves whose tangent measure is the uniform measure σ_{n-1}. For $n=2$ the circle of length L is a minimizer, and it is the unique minimizer among convex curves.*
4. *For $p=4$, there are infinitely many minimizers, given by curves whose tangent measure satisfies the isotropy condition (16). For $n=2$, among them there is the circle and all the regular polygons of perimeter L.*
5. *For $p>4$, minimizers are closed curves whose tangent measure is $\mu_{\rm sim}$. For $n=2$ equilateral triangles of perimeter L are minimizers, and they are the unique minimizers among convex curves.*

Proof. The statements follow directly from the characterization of $c(n,p)$ and the corresponding minimizers of J_p.

For $0 < p < 2$, by Proposition 4.4 the minimizer of J_p is $\mu = \frac{1}{2}(\delta_v + \delta_{-v})$, which corresponds to curves whose tangent vector takes only the two opposite values v and $-v$. Such curves are multiply covered segments in the direction v, with even multiplicity at every point.

The case $p = 2$ is trivial since J_2 is constant.

For $2 < p < 4$, Proposition 4.7 states that the unique minimizer of J_p is σ_{n-1}, so that minimizing curves must have $\mu_\gamma = \sigma_{n-1}$. For $n = 2$ this condition is satisfied by the circle of length L, for $n > 2$ there exist curves whose tangent vector is uniformly distributed on $\mathbb{S}^{n-1}$ (see Remark 5.9 below).

For $p = 4$ the result follows directly from Proposition 4.9.

For $p > 4$, Theorem 4.11 states that the unique minimizer of J_p is μ_{sim}, so that minimizing curves must have $\mu_\gamma = \mu_{\mathrm{sim}}$. For $n = 2$ this condition is satisfied by equilateral triangles of perimeter L. □

Remark 5.9. For $2 < p < 4$ and $n > 2$ minimizers are non-planar curves whose tangent vectors cover uniformly the unit sphere, and can be constructed by integrating Peano-type curves with values on $\mathbb{S}^{n-1}$ (see for instance [17, Chapter 3]). Notice that these curves can be of class $C^{1,\alpha}$ with $\alpha \leq \frac{1}{2}$, but not of class $C^{1,1}$.

In the following theorem we describe the closed curves maximizing (31). We omit the proof which is analogous to that of Theorem 5.8.

Theorem 5.10 (Maximizing curves)**.** *Let $L > 0$ be fixed. The maximum value of E_p among closed rectifiable curves of length L is $L^2\,C(n,p)$, where $C(n,p)$ is given by Theorem 4.4, Proposition 4.2, Theorem 4.7, Theorem 4.9, and Theorem 4.11. In particular,*

1. *For $0 < p < 2$, maximizers are curves whose tangent measure is the uniform measure σ_{n-1}. For $n = 2$ the circle of length L is a maximizer, and it is the unique maximizer among convex curves. The maximum value is $L^2 J_p(\sigma_{n-1})$.*
2. *For $p = 2$, every curve has energy $2L^2$ and is a maximizer.*
3. *For $p > 2$, the maximum is attained by curves that are multiply covered segments of length L (where the length is counted with multiplicity). The maximum value is $2^{p-1}L^2$.*

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