EXISTENCE OF A NON-STANDARD ISOPERIMETRIC TRIPLE PARTITION

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Abstract. We prove the existence of an isoperimetric 3-partition of $\mathbb{R}^8$ with one set of finite volume and two sets of infinite volume, which is asymptotic to a singular minimal cone.

Contents

1. Introduction

An M-*cluster* in $\mathbb{R}^d$ is an M-tuple $\mathbf{E} = (E_1, \dots, E_M)$ of pairwise disjoint subsets of $\mathbb{R}^d$. We say that $\mathbf{E}$ is an *isoperimetric cluster* if the surface area of its boundary $\mathcal{H}^{d-1}(\partial\mathbf{E}) = \mathcal{H}^{d-1}(\partial E_1 \cup \dots \cup \partial E_M)$ is minimal among all M-clusters $\mathbf{F}$ whose regions have the same volumes: $|F_k| = |E_k|$, $k = 1, \dots, M$. In the case $M = 1$, clusters are single sets and isoperimetric clusters are isoperimetric sets, i.e., balls.

Similarly, an N-*partition* is an N-tuple $\mathbf{E} = (E_1, \dots, E_N)$ of pairwise disjoint subsets $\mathbb{R}^d$ whose union is the entire space $\mathbb{R}^d$. If all but one of the regions are bounded, the interface $\partial\mathbf{E} = \partial E_1 \cup \dots \cup \partial E_N$ can have finite surface area, whilst if two regions are unbounded the total area must be infinite. Recently, in [1], the natural concept of *isoperimetric partition* is introduced: a partition $\mathbf{E}$ is *isoperimetric* if it is *locally isoperimetric* which means that $\mathcal{H}^{d-1}(\partial\mathbf{E} \cap B) \le \mathcal{H}^{d-1}(\partial\mathbf{F} \cap B)$ whenever B is bounded and $\mathbf{F}$ is an N-partition such that $|E_k| = |F_k|$ and $E_k \triangle F_k \Subset B$, for $k = 1, \dots, N$. If an N-partition has a single unbounded region, the bounded regions compose an M-cluster with $M = N - 1$ and the N-partition is isoperimetric if and only if the corresponding M-cluster is isoperimetric. On the other hand, if all the regions of an 2-partition are unbounded, the isoperimetric constraint on the

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volumes becomes empty (see [7]) so that isoperimetric 2-partitions correspond to *locally minimal sets.*

A lens partition $\mathbf{L} = (L_1, L_2, L_3)$ in $\mathbb{R}^d$ is a partition in which the bounded region L_1 is the intersection of two balls of equal radius R and centers $\mathbf{p}_1$ and $\mathbf{p}_2$ at distance R. The two unbounded regions L_2 and L_3 share a common boundary on the hyperplane equidistant from $\mathbf{p}_1$ and $\mathbf{p}_2$. In [1] it is shown that a 3-partition of $\mathbb{R}^2$ with a single bounded region is isoperimetric if and only if it is a lens partition. The analogous result in $\mathbb{R}^d$, for $d \leq 7$, was obtained in [17] (existence) and [7] (uniqueness), where the authors suggest that there might be a counterexample in $\mathbb{R}^8$, in analogy with what happens for locally minimal sets. In this paper, we provide such an example.

If $M \leq d+1$, $N = M+1$, there is a *standard* way to partition the sphere $\mathbb{S}^d$ into N equal regions. Take N mutually equidistant points $(\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_N) \in \mathbb{S}^d$, and consider the corresponding Voronoi partition of $\mathbb{S}^d$, i.e., the partition $\mathbf{V}$ with regions $V_k = \left\{\mathbf{x} \in \mathbb{S}^d \colon |\mathbf{x} - \mathbf{p}_k| \leq |\mathbf{x} - \mathbf{p}_j|, j = 1, \ldots, N\right\}$. The stereographic projection $\pi \colon \mathbb{S}^d \to \mathbb{R}^d \cup \{\infty\}$ maps the Voronoi partition V into a partition of $\mathbb{R}^d$. If we suppose that the north pole $\nu = \pi^{-1}(\infty) \in \mathbb{S}^d$ is an interior point of the region V_0, the regions $E_k = \pi(V_k)$ for $k = 1, \ldots, M = N-1$ are bounded and together form an M-cluster $\mathbf{E}$, which is called a *standard cluster.* On the other hand, if the north pole happens to lie on the boundary of two or more regions, the stereographic projection gives an N-partition of $\mathbb{R}^d$ which does not correspond to an M-cluster. We call these partitions *standard partitions.*

A long-standing conjecture by J. Sullivan states that, for $M \leq d+1$, isoperimetric M-clusters coincide with standard clusters. The conjecture was confirmed in the case $M = 2$, $d = 2$ by [9], in the case $M = 2$, $d = 3$ by [11], in the case $M = 2$, $d \geq 4$ by [20], and in the case $M = 3$, $d = 2$ by [22]. More recently, in [13], the conjecture has been confirmed in the cases $M = 3$, $d \geq 3$ and $M = 4$, $d \geq 4$. Moreover, in [15], the authors prove that standard N-partitions (for all $N \leq d+2$), and in particular standard M-clusters (for $M \leq d+1$), are *stable* if $d \geq 3$. In [18, 19], an example of an isoperimetric cluster with $d = 2$, $M = 4$, necessarily non-standard, is given. To our knowledge, there are no other proven examples of isoperimetric clusters.

In [17], we prove that any partition obtained as the limit of standard isoperimetric clusters is standard and isoperimetric. As a consequence, recalling the result in [14], we obtain that standard partitions are isoperimetric for all $N \leq \min(d+1, 5)$. In the case $N = 3$, $d \leq 7$ in [7] the authors prove a uniqueness result which ensures that all isoperimetric partitions are standard. If $d \geq 8$, there are examples (the first one being the Simons' cone in $d = 8$) of locally minimal sets in $\mathbb{R}^d$ which are singular cones: in our terminology, these are 2-partitions which are non-standard.

We also point out that in [5] there are examples of N-partitions of $\mathbb{R}^d$ which are isoperimetric but non-standard for $N = 2d \geq 8$ and $N = d \geq 6$, whose boundary is the cone over the $(d-1)$-skeleton of the hypercube.

In the present paper, we prove that also in the case $N = 3$, there exists a non-standard isoperimetric partition in dimension $d = 8$ (see Theorem 5.12), as was conjectured in [17] and in [7]. Even though the construction is based on the Simons' cone, we are not able to show that the non-standard isoperimetric partition we find is asymptotic to the Simons' cone. More generally, one could ask whether, given a singular minimal cone in $\mathbb{R}^d$, there exists an isoperimetric 3-partition asymptotic to that cone.

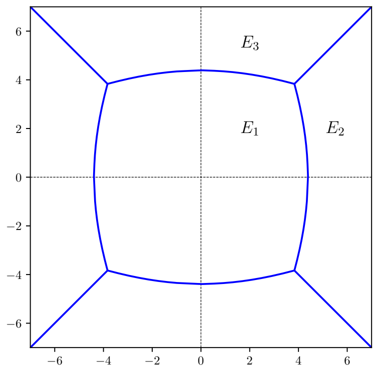

FIGURE 1. Numerically computed image of the conjectured isoperimetric 3-partition of $\mathbb{R}^8$. The actual partition in $\mathbb{R}^8$ would be the triple $(r(E_1), r(E_2), r(E_3))$, where $E_1, E_2, E_3 \subseteq \mathbb{R}^2$ are depicted in the figure, and $r(E) = \left\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^4 \times \mathbb{R}^4 \colon (|\mathbf{x}|, |\mathbf{y}|) \in E\right\}$. In this picture, where the mean curvature of the bounded region is 1, we have $\mathrm{Per}(r(E_1)) \approx 27.91 \cdot 10^5$, $|r(E_1)| \approx 16.04 \cdot 10^5$, while the perimeter of the missing cone inside $r(E_1)$ is $\approx 9.58 \cdot 10^5$, so the defect is ≈ 6.82. The defect of the *barrel* (i.e., the partition with a square in place of E_1) is ≈ 7.10, and the defect of the lens partition is ≈ 7.29 (see Lemma 5.6).

The idea of the proof is as follows. First, we consider the 3-partition obtained by adding the cylinder $\mathbb{B}^4 \times \mathbb{B}^4$ to the Simons' cone, and note that the perimeter gap required to pass from the Simons' cone to the barrel partition (which we call the *defect*) is less than the perimeter gap required to pass from a hyperplane to a lens partition (see Lemma 5.6). Then we consider the isoperimetric 3-partition which has the boundary data of the Simons' cone on a large sphere, and a small region of prescribed volume. The defect of such a partition is clearly smaller than the defect of the barrel partition. We then let the radius of the sphere go to infinity and apply concentration compactness to find a possibly infinite set of limit partitions. Next, we observe that if all these limit partitions were lenses, the total defect, eventually dominated by the defect of the isoperimetric partition, would be larger than the defect of a single lens. This means that at least one of the limit partitions must be different from a lens. Now we face the major problem in this approach: the fact that limit partitions could *see* the boundary of the large sphere as it goes to infinity. This means that we are forced to study isoperimetric partitions not only in the whole space $\mathbb{R}^8$, but also in a half-space which is the limit of the spheres.

Of course, the barrel partition is not isoperimetric (the triple junctions have an angle of 90 degrees). We can numerically find the deformation of the barrel that yields a partition with the same rotational symmetries, the correct angles, and constant mean curvature (see Figure 1). Proving that this partition is, in fact, isoperimetric would require some new ideas, because the tools used to prove the minimality of Simons' cone are not applicable to isoperimetric partitions.

In the development of our proof, we are led to consider isoperimetric partitions contained in a half-space, and we realized that, when $d \geq 4$, there could exist a 3-partition in a half-space which is not a lens partition.

The plan of the paper is as follows: in Section 2, we introduce some notation and give the definition of isoperimetric partition. In Section 3, we recall the concentration-compactness of partitions under natural bounds on the perimeter of the regions (see Theorem 3.1), and we prove the closure of isoperimetric partitions (see Theorem 3.6). In Section 4, we present the monotonicity formula for isoperimetric partitions in a cone (see Theorem 4.3) and the existence of blow-down partitions (see Corollary 4.4). Finally, in Section 5, we prove the existence of a non-standard isoperimetric 3-partition, asymptotic to a singular cone, which is the main result of this work (see Theorem 5.12).

After this paper was completed, we were informed by Lia Bronsard that an analogous result was obtained independently in [6].

2. Notation and preliminary definitions

If A, B are subsets of $\mathbb{R}^d$, we denote with $A \triangle B = (A \setminus B) \cup (B \setminus A)$ the symmetric difference between A and B.

For $B \subseteq \mathbb{R}^d$ we denote with $\bar{B}$ and ∂B the topological closure and boundary of B. We denote with $\mathbb{B}^d = \left\{\mathbf{x} \in \mathbb{R}^d \colon\ |\mathbf{x}| < 1\right\}$ the unit ball of $\mathbb{R}^d$ and let $\mathbb{S}^{d-1} = \partial \mathbb{B}^d$ be its boundary. We let $\omega_d = \left|\mathbb{B}^d\right|$ be the measure of the unit ball. We denote with $B_\rho(\mathbf{x}) := \mathbf{x} + \rho \mathbb{B}^d$ the ball of radius $\rho > 0$ centered at $x \in \mathbb{R}^d$, and let $B_\rho := B_\rho(\mathbf{0})$.

A set $C \subseteq \mathbb{R}^d$ is said to be a *cone* if for every $t > 0$ and $\mathbf{x} \in C$ we have $t\mathbf{x} \in C$ (a positively homogeneous cone with vertex in the origin). A *hyperplane* of $\mathbb{R}^d$ is a $(d-1)$-dimensional vector subspace of $\mathbb{R}^d$ (passing through the origin). We denote with $(\mathbf{x}, \mathbf{y})$ the scalar product of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$. An *half-space* of $\mathbb{R}^d$ is any set of the form $\left\{\mathbf{x} \in \mathbb{R}^d \colon (\mathbf{x}, \nu) > 0\right\}$ for some $\nu \in \mathbb{R}^d$. Hyperplanes and half-spaces are cones. The translation of a hyperplane (resp. half-space) will be called affine hyperplane (resp. half-space).

$|E|$ will denote the Lebesgue measure of a measurable set $E \subseteq \mathbb{R}^d$. Given E, E_n measurable subsets of $\mathbb{R}^d$ we will write $E_n \xrightarrow{L^1} E$ if $|E_n \triangle E| \to 0$ as $n \to +\infty$ and $E_n \xrightarrow{L^1_{\rm loc}} E$ if $|(E_n \triangle E) \cap B_r| \to 0$ for all $r > 0$. We also consider on $\mathbb{R}^d$ the usual Hausdorff, k-dimensional, $k \leq d$, measures $\mathcal{H}^h$. If $E \subseteq \mathbb{R}^d$ is a measurable set, and $B \subseteq \mathbb{R}^d$ is a Borel set, we define the *perimeter* of E in B as:

$$\operatorname{Per}(E, B) := \|D 1_E\| (B),$$

where $\|D1_E\|$ is the total variation of the distributional derivative of the characteristic function of E. We let $\operatorname{Per}(E) := \operatorname{Per}(E, \mathbb{R}^d)$ be the *perimeter* of E. We say that a measurable set $E \subseteq \mathbb{R}^d$ has *finite perimeter* if $\operatorname{Per}(E) < +\infty$, we say that E is a *Caccioppoli set* if $\operatorname{Per}(E, B_r) < +\infty$ for all $r > 0$. In this case $\operatorname{Per}(E, \cdot)$ is a measure which coincides with $\mathcal{H}^{d-1} \llcorner \partial^* E$ where $\partial^* E$ is the *reduced boundary* of E, i.e. the set of points of ∂E where an approximate normal vector can be defined.

$H^{(t)}$ denotes the sets of points of H of density t in H; when H is a Caccioppoli set, ν_H denotes the measure theoretic exterior normal, $\partial^e H$ the essential boundary of H, so that for any $F \in (C^1_c(\mathbb{R}^d))^d$, we have

$$\partial^* H \subseteq H^{(\frac{1}{2})} \subseteq \partial^e H, \quad \mathcal{H}^{d-1}(\partial^e H \setminus \partial^* H) = 0, \quad \int_H \operatorname{div} F \, d\mathcal{H}^d = \int_{\partial^e H} \langle F \cdot \nu_H \rangle \, d\mathcal{H}^{d-1}.$$

Definition 2.1. For $N \in \mathbb{N}$ let $E_1, \dots, E_N$ be measurable subsets of $\mathbb{R}^d$. The N-uple $\mathbf{E} = (E_1, \dots, E_N)$ is said to be an *N-partition* of $\mathbb{R}^d$ if $|E_j \cap E_k| = 0$ when $j \neq k$ and

$$\left| \mathbb{R}^d \setminus \bigcup_{j=1}^N E_j \right| = 0.$$

If each E_j is a Caccioppoli set (has locally finite perimeter) we say that $\mathbf{E}$ is a *Caccioppoli partition.*

We say that the N-partition $\mathbf{E}$ is a *proper* partition if $|E_j| > 0$ for all $j = 1, \dots, N$, otherwise we might specify that it is *improper.*

We define

$$\mathbf{m}(\mathbf{E}) := (|E_1|, \dots, |E_N|) \in [0, +\infty]^d$$

and

$$\mathrm{Per}(\mathbf{E}, B) := \frac{1}{2} \sum_{i=1}^N \mathrm{Per}(E_k, B) \in [0, +\infty].$$

We say that a sequence of N-partitions $\mathbf{E}_n$ converges to an N-partition $\mathbf{F}$ in L^1_{loc} if for all measurable bounded $B \subseteq \mathbb{R}^d$ for all $j = 1, \dots, N$ we have

$$\left| (E_j^n \triangle F_j) \cap B \right| \to 0.$$

Operators on N-uples of sets are intended componentwise, as one would understand with usual vectors. For example, if $\mathbf{E}$ and $\mathbf{F}$ are N-partitions in $\mathbb{R}^d$, $A \subseteq \mathbb{R}^d$ and $\mathbf{x} \in \mathbb{R}^d$ we adopt the following notation

$$\begin{aligned} \mathbf{E} \cup A &:= (E_1 \cup A, \dots E_N \cup A), \\ \mathbf{E} \cup \mathbf{F} &:= (E_1 \cup F_1, \dots, E_N \cup F_N), \\ \mathbf{E} + \mathbf{x} &:= (E_1 + \mathbf{x}, \dots, E_n + \mathbf{x}), \end{aligned}$$

where

$$E_k + \mathbf{x} = \{\mathbf{y} + \mathbf{x} \colon \mathbf{y} \in E_k\}.$$

Notice that $\mathbf{E} \cup \mathbf{F}$ and $\mathbf{E} \cup A$ in general are not partitions.

Definition 2.2 (isoperimetric partitions)**.** Let $X \subseteq \mathbb{R}^d$ be a closed set and $\mathbf{E}$ be an N-partition *of* $\mathbb{R}^d$, $J \subseteq \{1, \dots, N\}$. We say that $\mathbf{E}$ is *J-isoperimetric in X* (or simply *J-isoperimetric* if $X = \mathbb{R}^d$) if for every compact set $K \subseteq X$ and every partition $\mathbf{F}$ of $\mathbb{R}^d$ such that

$$\mathbf{F} \triangle \mathbf{E} \subseteq K \tag{1}$$

$$|F_j \cap K| = |E_j \cap K| \qquad \text{for all } j \in J \tag{2}$$

one has

$$\mathrm{Per}(\mathbf{E}, K) \leq \mathrm{Per}(\mathbf{F}, K).$$

We say that the partition $\mathbf{E}$ is *isoperimetric in X* (or simply *isoperimetric* if $X = \mathbb{R}^d$) if $\mathbf{E}$ is J-isoperimetric in X for $J = \{j \in \{1, \dots, N\} : |E_j| < +\infty\}$.

Remark 2.3. If $J = \{j \colon |E_j| < +\infty\}$ and (1) holds then (2) is equivalent to

$$|F_j| = |E_j| \qquad \text{for all } j = 1, \dots, N. \tag{3}$$

Remark 2.4. In the case when $X = \mathbb{R}^d$ and the partition has a single infinite region, by dropping the infinite region we obtain what is usually called a *cluster* and isoperimetric partitions correspond to *isoperimetric clusters* (see [17, Definition 2.3 and Proposition 2.11]).

When $X = \mathbb{R}^d$, the notion of J-isoperimetric partition given above closely resembles that of *locally minimizing* $(N, M)-$*clusters* given in [7, Definition 2.5].

3. Compactness of partitions

We extend to partitions of $\mathbb{R}^d$ the results proven in [16, Theorems 3.3, 4.2, Proposition 5.2] for clusters (i.e. partitions with only one region with infinite measure) in a more abstract setting. In this section we only prove the statements concerning partitions, while some known results are collected in Appendix A for the reader's convenience.

Theorem 3.1. *Let $\mathbf{E}_n$ be a sequence of N-partitions of $\mathbb{R}^d$, and let*

$$m_j := \limsup_n \left|E_j^n\right| \in [0, +\infty], \qquad j = 1, \dots, N.$$

Assume that

$$\sup_{x\in\mathbb{R}^d} \sup_{n\in\mathbb{N}} \operatorname{Per}(\mathbf{E}_n, B_1(x)) < +\infty,$$
$$m_j < +\infty \implies \sup_{n\in\mathbb{N}} \operatorname{Per}(E_j^n) < +\infty.$$

Then there exists a subsequence of $\mathbf{E}_n$, non-relabeled, an N-partition $\mathbf{F}$ of $(\mathbb{R}^d)^{\mathbb{N}}$ (i.e. $\mathbf{F} = (\mathbf{F}^i)_{i\in\mathbb{N}}$, $\mathbf{F}^i$ is an N-partition of $\mathbb{R}^d$) and $\mathbf{x}_n^i \in \mathbb{R}^d$ such that

$$\lim_{n\to+\infty} \left|\mathbf{x}_n^i - \mathbf{x}_n^j\right| \to \infty \qquad \text{if } i \neq j, \tag{4}$$

$$\mathbf{E}_n - \mathbf{x}_n^i \to \mathbf{F}^i \qquad \text{in } L^1_{\text{loc}}(\mathbb{R}^d) \text{ for each } i \in \mathbb{N}, \tag{5}$$

$$\sum_{i\in\mathbb{N}} \left|F_j^i\right| = \lim_n \left|E_j^n\right|, \quad \text{if } m_j < +\infty. \tag{6}$$

Proof. Up to subsequences we assume that for each $j \in \{1, \dots, N\}$ there exists $\lim_n \left|E_j^n\right| = m_j$. Let $J := \{j = 1, \dots, N : m_j < +\infty\}$ and consider the sequence of sets

$$E^n := \bigcup_{j\in J} E_j^n.$$

Applying Theorem A.3 to the sequence E^n, up to a subsequence, we can find $\mathbf{x}_n^i$ and F^i satisfying (84), (85) and (86) *i.e.*

$$\sum_i \left|F^i\right| = \lim_n \sum_{j\in J} \left|E_j^n\right|. \tag{7}$$

In view of Lemma A.1, up to a subsequence, we can assume that there exist F_j^i such that for all $j = 1, \dots, N$ we have $E_j^n - \mathbf{x}_n^i \to F_j^i$ in L^1_{loc}. For each i the N-uple $\mathbf{F}^i := (F_1^i, \dots, F_N^i)$ is a partition of $\mathbb{R}^d$. In particular, up to a negligible set, we have $F^i = \bigcup_{j\in J} F_j^i$ so obtaining from (7)

$$\sum_i \sum_{j\in J} \left|F_j^i\right| = \lim_n \sum_{j\in J} \left|E_j^n\right| = \lim_n \left|E^n\right|. \tag{8}$$

Consider any $R > 0$ and let n be large enough so that $B_R(\mathbf{x}_n^j)$ are pairwise disjoint. By the semicontinuity of Lebesgue measure applied to $E_j^n - \mathbf{x}_n^i \to F_j^i$, for all $j \in \{1, \dots, N\}$, we have

$$\sum_{i=0}^{M} \left|F_j^i \cap B_R\right| = \lim_n \sum_{i=0}^{M} \left|E_j^n \cap B_R(\mathbf{x}_n^i)\right| = \lim_n \left|E_j^n \cap \bigcup_{i=0}^{M} B_R(\mathbf{x}_n^i)\right| \le \liminf_n \left|E_j^n\right|$$

hence, for $R \to \infty$ and then $M \to \infty$:

$$\sum_i \left|F_j^i\right| \le \lim_M \sup_R \sum_{i=0}^{M} \left|F_j^i \cap B_R\right| \le \liminf_n \left|E_j^n\right| \le m_j, \quad \text{for all } j. \tag{9}$$

Finally using, for $j \in J$, (8), and (9),

$$\begin{aligned}\lim_n \sum_{j\in J} \left|E_j^n\right| &= \lim_n |E^n| = \sum_i \left|F^i\right| = \sum_{j\in J} \sum_i \left|F_j^i\right| \\ &\le \sum_{j\in J} \liminf_n \left|E_j^n\right| \le \liminf_n \sum_{j\in J} \left|E_j^n\right|.\end{aligned}$$

Since the previous inequality is actually an equality, also in (9) we must have an equality for all $j \in J$, and we obtain (6). □

Remark 3.2. By semicontinuity of the perimeter, one also gets for all open sets B

$$\sum_{i\in\mathbb{N}} \operatorname{Per}(F_j^i, B) \le \liminf_n \operatorname{Per}(E_n^j, \bigcup_i (B + \mathbf{x}_n^i)), \quad \text{for all } j, \tag{10}$$

Lemma 3.3. *Let $\mathbf{E}$ and $\mathbf{F}_i$ be Caccioppoli N-partitions in $\mathbb{R}^d$ for $i = 1, \dots, n$. Let $R > 0$ and $x_i \in \mathbb{R}^d$, for $i = 1, \dots, N$, be such that the balls $B_R(x_i)$ are pairwise disjoint. Let B any other ball or $\mathbb{R}^d$. For all $\rho \le R$ define*

$$\mathbf{G}_\rho = (\mathbf{E} \setminus \bigcup_{i=1}^{n} B_\rho(x_i) \cap B) \cup \bigcup_{i=1}^{n} (\mathbf{F}^i \cap B_\rho(x_i) \cap B).$$

Then the set of $\rho \in (r, R)$ such that

$$\sum_{i=1}^{n} \operatorname{Per}(\mathbf{G}_\rho(x_i), (\partial B_\rho) \cap B) \le \sum_{i=1}^{n} \frac{\left\|\mathbf{m}((\mathbf{E}\triangle\mathbf{F}_i) \cap (B_R(x_i) \setminus B_r(x_i)) \cap B)\right\|_1}{2 \cdot (R - r)} \tag{11}$$

has positive measure (where $\|v\|_1 = \sum_{k=1}^d |v_k|$ is the ℓ_1-norm of the vector $v \in \mathbb{R}^d$).

Proof. For all $0 < r < R$, for all $i = 1, \dots, n$, for all $k = 1, \dots, n$ one has (see [17, Lemma 2.6]):

$$\begin{aligned}&\int_r^R \operatorname{Per}(G_\rho^k, (\partial B_\rho(x_i)) \cap B)\, d\rho \\ &= \left|(E^k \triangle F_i^k) \cap (B_R(x_i) \setminus B_r(x_i)) \cap B\right|.\end{aligned}$$

Summing up on $k = 1, \dots, n$, on $i = 1, \dots, N$, averaging on $\rho \in (r, R)$, and dividing by two, we obtain

$$\fint_r^R \sum_{i=1}^{n} \operatorname{Per}(\mathbf{G}_\rho, (\partial B_\rho(x_i)) \cap B)\, d\rho = \sum_{i=1}^{n} \frac{\left\|\mathbf{m}((\mathbf{E}\triangle\mathbf{F}_i) \cap (B_R(x_i) \setminus B_r(x_i)) \cap B)\right\|_1}{2 \cdot (R - r)}.$$

The conclusion follows by noting that the integrand on the left-hand side cannot be larger than its average. □

We recall the following result from [17, Theorem 2.8].

Theorem 3.4. *Let Ω be an open subset of $\mathbb{R}^d$. Let $\mathbf{F} = (F_1, \dots, F_N)$ be a (possibly improper) N-partition of Ω. For all $k = 1, \dots, N$ with $|F_k| > 0$, let $x_k \in \Omega$, and $\rho_k > 0$, be given so that the balls $B_{\rho_k}(x_k)$ are contained in Ω, are pairwise disjoint and $|F_k \cap B_r(x_k)| > \frac{1}{2}\omega_d r^d$ for all $r \leq \rho_k$.*

Let $A = \bigcup_k B_{\rho_k}(x_k)$. Then for every $\mathbf{a} = (a_0, \dots, a_N)$ with $a_k \geq -|F_k \cap B_{\rho_k}(x_k)|$ when $|F_k| > 0$, $a_k \geq 0$ when $|F_k| = 0$, and $\sum_{k=1}^N a_k = 0$, there exists a partition $\mathbf{F}' = (F'_1, \dots, F'_N)$ such that for all $k = 1, \dots, N$:

(i) $F'_k \triangle F_k \subseteq A$,

(ii) $|F'_k \cap A| = |F_k \cap A| + a_k$,

(iii) $$P(F'_k, \bar{A}) \leq P(F_k, \bar{A}) + C_1 \cdot \sum_{j=1}^N |a_j|^{1-\frac{1}{d}},$$

with $C_1 = C_1(d, N)$ not depending on $\mathbf{F}$.

Definition 3.5. Let $N \in \mathbb{N}$, $J \subseteq \{1, \dots, N\}$ and for all $i \in \mathbb{N}$ let $\bar{\Pi}^i$ be closed subsets of $\mathbb{R}^d$. We say that the N-partition $\mathbf{F} = (\mathbf{F}^i)_{i\in\mathbb{N}}$ of $(\mathbb{R}^d)^{\mathbb{N}}$ is a J-isoperimetric N-partition in $X = \bigotimes_{i\in\mathbb{N}} \bar{\Pi}^i \subseteq (\mathbb{R}^d)^{\mathbb{N}}$ if for all compact $K \subseteq \mathbb{R}^d$ and for all N-partition $\mathbf{G} = (\mathbf{G}^i)_{i\in\mathbb{N}}$ of $(\mathbb{R}^d)^{\mathbb{N}}$ such that

$$\left\{i \in \mathbb{N} \colon \mathbf{F}^i \neq \mathbf{G}^i\right\} \text{ is finite,} \tag{12}$$

$$\mathbf{F}^i \triangle \mathbf{G}^i \subseteq K \cap \bar{\Pi}^i \qquad \text{for all } i \in \mathbb{N}, \tag{13}$$

$$\sum_i \left|G^i_j \cap K \cap \bar{\Pi}^i\right| = \sum_i \left|F^i_j \cap K \cap \bar{\Pi}^i\right| \qquad \text{for all } j \in J, \tag{14}$$

one has

$$\sum_i \operatorname{Per}(\mathbf{F}^i, K \cap \bar{\Pi}^i) \leq \sum_i \operatorname{Per}(\mathbf{G}^i, K \cap \bar{\Pi}^i). \tag{15}$$

The following result extends [17, Theorem 2.13] to the case of conical boundary data given on closed balls invading the space.

Theorem 3.6 (closure of isoperimetric partitions with boundary data)**.** *Let $\mathbf{S}$ be a Caccioppoli N-partition of $\mathbb{R}^d$ which is a cone, i.e. $\mathbf{S} = \mathbb{R}_+ \cdot \mathbf{S}_0$ with $\mathbf{S}_0 = \mathbf{S} \cap \partial B_1$. Suppose that*

$$\theta := \sup_{\mathbf{x}\in\mathbb{R}^d} \sup_{\rho>0} \frac{\operatorname{Per}(\mathbf{S}, B_\rho(\mathbf{x}))}{\rho^{d-1}} < +\infty. \tag{16}$$

Let $\mathbf{E}_n = ((E_n)_1, \dots, (E_n)_N)$ be a sequence of N-partitions of $\mathbb{R}^d$ which are J-isoperimetric in $\bar{B}_{R_n} \subseteq \mathbb{R}^d$ for some $J \subseteq \{1, \dots, N\}$ and $R_n \to +\infty$, such that $\mathbf{E}_n \setminus \bar{B}_{R_n} = \mathbf{S} \setminus \bar{B}_{R_n}$.

For $i \in \mathbb{N}$, suppose $\mathbf{F}^i$ are N-partitions of $\mathbb{R}^d$, and $\mathbf{x}^i_n \in \mathbb{R}^d$, satisfy

$$\lim_{n\to+\infty} \left|\mathbf{x}^i_n - \mathbf{x}^j_n\right| \to \infty \qquad \textit{if } i \neq j, \tag{17}$$

$$\mathbf{E}_n - \mathbf{x}^i_n \to \mathbf{F}^i \qquad \textit{in } L^1_{\text{loc}}(\mathbb{R}^d) \textit{ for each } i \in \mathbb{N}. \tag{18}$$

Moreover assume that, for all $i \in \mathbb{N}$, we have, as $n \to +\infty$:

$$B_{R_n} - \mathbf{x}^i_n \xrightarrow{L^1_{\text{loc}}} \Pi^i \tag{19}$$

where Π^i is either the whole space $\mathbb{R}^d$ or it is an open affine half-space of $\mathbb{R}^d$, and

$$\mathbf{S} - \mathbf{x}_n^i \xrightarrow{L^1_{\mathrm{loc}}} \mathbf{T}^i, \tag{20}$$

$$S_k - \mathbf{x}_n^i \text{ locally Hausdorff converge to } T_k^i, \tag{21}$$

where $\mathbf{T}^i$ is a N-partition of $\mathbb{R}^d$. Moreover suppose that for all $i \in \mathbb{N}$ and all open bounded $B \subseteq \mathbb{R}^d$

$$\mathrm{Per}(\mathbf{T}^i, \partial B) = 0 \implies \lim_{n\to+\infty} \mathrm{Per}(\mathbf{S} - \mathbf{x}_n^i, B) = \mathrm{Per}(\mathbf{T}^i, B) \tag{22}$$

$$\mathrm{Per}(\mathbf{T}^i, B) = \mathcal{H}^{d-1}(B \cap \partial \mathbf{T}^i). \tag{23}$$

Then $\mathbf{F} = (\mathbf{F}^i)_{i\in\mathbb{N}}$ is a J-isoperimetric N-partition in $X = \bigotimes_{i\in\mathbb{N}} \bar{\Pi}^i \subseteq (\mathbb{R}^d)^{\mathbb{N}}$ (see Definition 3.5). In particular each $\mathbf{F}^i$ is a J-isoperimetric partition in $\bar{\Pi}^i$ for all $i \in \mathbb{N}$.

Proof. Note that if $\left|\mathbf{x}_n^i\right| - R_n \to +\infty$ we would have $\Pi^i = \emptyset$, and this case has been excluded in the statement because it is not relevant. If $\left|\mathbf{x}_n^i\right| - R_n \to -\infty$ we have $\Pi^i = \mathbb{R}^d$: in this case the construction below is easier. So, our focus is in the case when Π^i is a half-space of $\mathbb{R}^d$.

In the case when Π^i is a half-space with outer normal ν_i, by (18) and (19) passing to the limit in $\mathbf{E}_n \setminus \bar{B}_{R_n} = \mathbf{S} \setminus \bar{B}_{R_n}$ we obtain that $\mathbf{F}^i \setminus \bar{\Pi}^i = \mathbf{T}^i \setminus \bar{\Pi}^i$. Since $B_{R_n}(-\mathbf{x}_n^i) \to \Pi^i$ in L^1_{loc}, one can check that

$$\nu^i = \lim_n \frac{\mathbf{x}_n^i}{|\mathbf{x}_n^i|}.$$

On the other hand, given $\mathbf{x} \in \mathbb{R}^d$ one has $\frac{\mathbf{x}+\mathbf{x}_n^i}{|\mathbf{x}+\mathbf{x}_n^i|} \to \nu^i$ locally uniformly as $n \to +\infty$, so that since $\mathbf{S} - \mathbf{x}_n^i$ converges locally to $\mathbf{T}^i$, being $\mathbf{S}$ a cone implies that $\mathbf{T}^i$ is equivalent to a cylinder perpendicular to $\partial\Pi^i$:

$$\partial \mathbf{T}^i = \partial \mathbf{T}_0^i + \mathbb{R}\nu^i$$

where $\mathbf{T}_0$ is the partition on $\partial \Pi^i$ which is the trace of $\mathbf{T}$. In particular, $\mathrm{Per}(\mathbf{T}^i, \partial\Pi^i + t\nu^i) = 0$ for all $t \in \mathbb{R}$. From (16) we obtain

$$\mathrm{Per}(\mathbf{T}^i, B_\rho(\mathbf{x})) \le \theta\rho^{d-1}, \qquad \forall \mathbf{x} \in \mathbb{R}^d, \rho > 0$$

whence, using (23), considering $\partial\mathbf{T}_0$ a partition in dimension $d-1$,

$$\mathcal{H}^{d-2}(\partial \mathbf{T}_0 \cap B_\rho(\mathbf{x})) \le \theta\rho^{d-2}, \qquad \forall \mathbf{x} \in \partial\Pi^i, \forall \rho > 0. \tag{24}$$

Given $\mathbf{x} \in B_R \cap \partial B_{R_n}(-\mathbf{x}_n^i)$, the normal vector to $B_{R_n}(-\mathbf{x}_n^i)$ in $\mathbf{x}$ is $\frac{\mathbf{x}+\mathbf{x}_n^i}{|\mathbf{x}+\mathbf{x}_n^i|}$ which is uniformly converging, in B_R, to ν^i as $n \to +\infty$. So, for n large enough, we can assume that the angle between the normal to ∂B_{R_n} and ν_i is small enough so that the orthogonal projection $\pi \colon \partial B_{R_n}(-\mathbf{x}_n^i) \to \partial \Pi^i$, if restricted to B_R, is injective and its inverse function is 2-Lipschitz. This means that for all measurable sets X which are cylinders in direction ν^i, one has

$$\mathcal{H}^{d-1}(X \cap B_R \cap \partial B_{R_n}(-\mathbf{x}_n^i)) \le 2\mathcal{H}^{d-1}(X \cap B_R \cap \partial\Pi^i). \tag{25}$$

Let $K \subseteq \mathbb{R}^d$ be a compact set, let $\mathbf{G} = (\mathbf{G}^i)_{i\in\mathbb{N}}$ be a partition satisfying (12), (13), (14), and let $\varepsilon > 0$ be small enough so that if $C_1 = C_1(d, N)$ is the constant given by Theorem 3.4, we have

$$\varepsilon < 1, \qquad \varepsilon < (NC_1)^2. \tag{26}$$

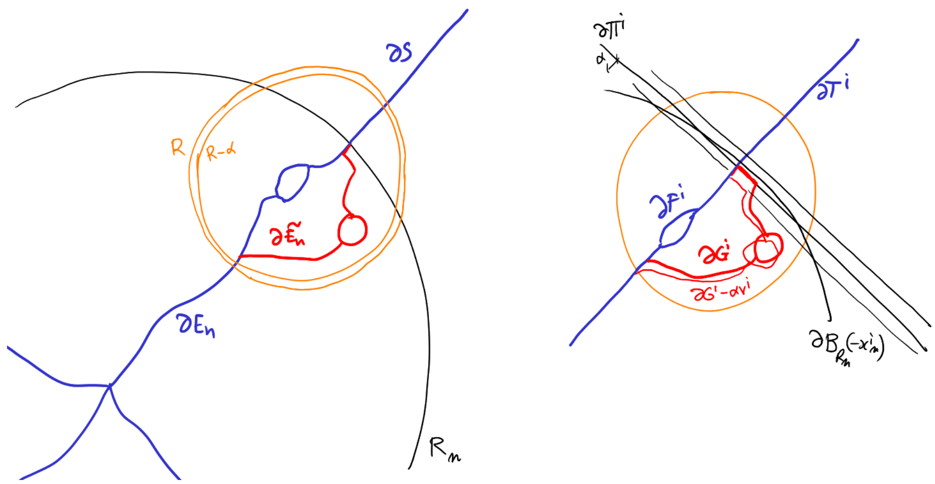

FIGURE 2. A reference picture for the proof of Theorem 3.6.

Our aim is to prove that the inequality (15) is valid up to an error of order ε. This will be obtained by building a cluster $\hat{\mathbf{E}}_n$ competing to the minimality of $\mathbf{E}_n$ for some large n. First a cluster $\tilde{\mathbf{E}}_n$ is obtained by patching all the given clusters $\mathbf{G}^i$ into large holes $\Omega^i_n = B_{R-\rho}(\mathbf{x}^i_n) \setminus B_{R_n}$ cut from $\mathbf{E}_n$. The radius $R > 0$ is large enough to contain all the variations $\mathbf{G}^i \triangle \mathbf{F}^i$ while n will be taken large enough so that the balls $B_R(\mathbf{x}^i_n)$ will be disjoint. However these balls in general can contain large portions of the boundary ∂B_{R_n} where we are required to keep the original boundary data of our sequence $\mathbf{E}_n$. To facilitate the matching of the boundary, the cluster $\mathbf{G}^i$ will be slightly translated inside the container Π^i so that the unknown behaviour of $\mathbf{G}^i$ on the boundary is replaced by $\mathbf{T}^i$ which can be assumed to be controlled. The parameter α will measure the entity of the translation. By taking n large enough we will ensure that α is larger than the distance, in B_R, between the translation of the curved boundary $\partial B_{R_n}(-\mathbf{x}^i_n)$ and its limit $\partial \Pi^i$. The cutting radius R must be slightly adjusted by a small quantity ρ to assure we don't have perimeter concentrating on ∂B_R. A small amount of the mass of the regions with prescribed measure of $\mathbf{E}_n$ can remain outside all the balls B_{R_n}, we are hence forced to slightly adjust the measure of the regions of $\mathbf{G}^i$, using the volume fixing Lemma, to recover the constraint and obtain a competitor $\hat{\mathbf{E}}_n$ to $\mathbf{E}_n$. In preparation to this we have to fix the small balls $B_{r^i}(\mathbf{y}^i_k)$ inside the regions of $\bar{G}^i$ where the mass will be exchanged as needed, and we must take care of ensuring that these balls remain in the patching area. The parameters $\varepsilon, R, \alpha, n, \rho$ will be chosen in this order. All their properties will be declared at once when they are introduced, to make it clear that there is no circular dependency among them.

Let $I = \{i \in \mathbb{N} \colon \mathbf{F}^i \neq \mathbf{G}^i\}$, I is a finite set in view of (12). Let $R > 0$ be large enough so that $K \subseteq B_{R-1}$, and $\mathrm{Per}(\mathbf{G}^i, \partial B_R))$.

In view of applying Theorem 3.4 to G^i, for every $i \in I$ consider points $\mathbf{y}^i_1, \dots, \mathbf{y}^i_N$ and a radius $r_i > 0$ such that $B_{r_i}(\mathbf{y}^i_k) \Subset \Pi^i \cap B_{R-1}$, and,

$$\left|G^i_k \cap \Pi^i \cap B_{R-1}\right| > 0 \implies \forall \rho < r_i \colon \left|G^i_k \cap B_\rho(\mathbf{y}^i_k)\right| > \frac{1}{2}\omega_d \rho^d.$$

This can be done by taking $\mathbf{y}^i$ a point of density 1 in $G_k^i \cap \Pi^i \cap B_{R-1}$ and r_i small enough.

For all Π^i which are half-spaces, let ν^i be the normal exterior versor, and set $\nu^i := 0$ when $\Pi^i = \mathbb{R}^d$. Let $\alpha \in (0, \frac{1}{2})$ be sufficiently small so that, being $C_1 = C_1(d, N)$ as in Theorem 3.4, we have

$$\sup_{\rho<\alpha} \left\|\mathbf{m}(\mathbf{G}^i \triangle (\mathbf{G}^i - \rho\nu^i))\right\|_1 \le \frac{1}{2}\left(\frac{\varepsilon}{N^2 C_1}\right)^{\frac{d}{d-1}} \le \varepsilon, \tag{27}$$

$$\sup_{\rho<\alpha} \left\|\mathbf{m}(\mathbf{F}^i \triangle (\mathbf{F}^i - \rho\nu^i))\right\|_1 \le \varepsilon, \tag{28}$$

$$\mathrm{Per}(\mathbf{G}^i, B_{R+\alpha} \cap (\Pi^i + 2\alpha\nu^i)) - \mathrm{Per}(\mathbf{G}^i, B_{R-\alpha} \cap (\Pi^i + 2\alpha\nu^i) \le \varepsilon, \tag{29}$$

$$B_{r_i}(\mathbf{y}_k^i) \Subset \Pi^i - \alpha\nu^i, \tag{30}$$

$$\mathrm{Per}(\mathbf{T}^i, B_R \cap (\bar\Pi^i + 2\alpha\nu^i) \setminus (\bar\Pi^i - 2\alpha\nu^i)) \le 4\alpha\theta R^{d-2} \le \varepsilon, \tag{31}$$

$$\mathrm{Per}(\mathbf{T}^i, B_R \setminus (\Pi^i + 2\alpha\nu^i)) \le \mathrm{Per}(\mathbf{T}^i, B_{R-\alpha} \setminus (\Pi^i + 2\alpha\nu^i)) + \varepsilon, \tag{32}$$

$$\mathrm{Per}(\mathbf{E}_n, B_R(\mathbf{x}_n^i)) \le \mathrm{Per}(\mathbf{E}_n, B_{R-\alpha}(\mathbf{x}_n^i)) + \varepsilon. \tag{33}$$

Take n large enough so that, using (17), (18), and (19), we have

$$R_n > R, \quad \left|\mathbf{x}_{n'}^i - \mathbf{x}_{n'}^\ell\right| > 3R \qquad \text{if } i, \ell \in I,\ i \ne \ell,\ n' \ge n, \tag{34}$$

$$\left\|\mathbf{m}(((\mathbf{E}_n - \mathbf{x}_n^i) \triangle \mathbf{F}^i) \cap B_R)\right\| < \frac{1}{2}\left(\frac{\varepsilon}{N^2 C_1}\right)^{\frac{d}{d-1}} \qquad \forall i \in I, \tag{35}$$

$$(\bar\Pi^i - \alpha\nu^i) \cap B_R \subseteq B_{R_n}(-\mathbf{x}_n^i) \cap B_R, \tag{36}$$

$$\bar B_{R_n}(-\mathbf{x}_n^i) \subseteq (\Pi^i + \alpha\nu^i), \tag{37}$$

$$\mathrm{Per}(\mathbf{F}^i, B_R) \le \mathrm{Per}(\mathbf{E}_n, B_R(\mathbf{x}_n^i)) + \varepsilon, \tag{38}$$

$$\mathrm{Per}(\mathbf{S} - \mathbf{x}_n^i, B_R \setminus (\bar\Pi^i - \alpha\nu^i)) \le \mathrm{Per}(\mathbf{T}^i, B_R \setminus (\bar\Pi^i - \alpha\nu^i)) + \varepsilon, \tag{39}$$

$$B_R \cap (S_k - \mathbf{x}_n^i) \subseteq (T_k^i)_\delta, \qquad \text{with } \delta = \frac{\varepsilon}{4\theta R^{d-2}}. \tag{40}$$

For some fixed $\rho \in (0, \alpha)$ consider $\Omega_n^i := B_{R-\rho}(\mathbf{x}_n^i) \cap B_{R_n}$, define $\Omega_n := \bigcup_i \Omega_n^i$, and

$$\tilde{\mathbf{E}}_n := [\mathbf{E}_n \setminus \Omega_n] \cup \bigcup_{i \in I} \left[(\mathbf{G}^i - \alpha\nu^i + \mathbf{x}_n^i) \cap \Omega_n^i\right]. \tag{41}$$

Clearly $\tilde{\mathbf{E}}_n \triangle \mathbf{E}_n \subseteq \bar{\Omega}_n \subseteq \bar{B}_{R_n}$ moreover by slightly adjusting $\rho = \rho_n \in (0, \alpha)$, in view of Lemma 3.3, we can suppose that

$$\begin{aligned} &\operatorname{Per}(\tilde{\mathbf{E}}_n, \partial B_{R-\rho}(\mathbf{x}_n^i) \cap B_{R_n}) \\ &\quad \le \frac{1}{2} \sum_{i \in I} \left\| \mathbf{m}((\mathbf{E}_n - \mathbf{x}_n^i) \triangle (\mathbf{G}^i - \alpha \nu^i) \cap B_R \setminus B_{R-1}) \right\|_1 \\ &\quad = \sum_{i \in I} \left\| \mathbf{m}((\mathbf{E}_n - \mathbf{x}_n^i) \triangle (\mathbf{F}^i - \alpha \nu^i) \cap B_R \setminus B_{R-1}) \right\|_1 \\ &\quad \le \sum_{i \in I} \left\| \mathbf{m}((\mathbf{E}_n - \mathbf{x}_n^i) \triangle (\mathbf{F}^i - \alpha \nu^i) \cap B_R) \right\|_1 \\ &\quad \le \sum_{i \in I} \left(\left\| \mathbf{m}((\mathbf{E}_n - \mathbf{x}_n^i) \triangle \mathbf{F}^i \cap B_R) \right\|_1 + \left\| \mathbf{m}(\mathbf{F}^i \triangle (\mathbf{F}^i - \alpha \nu^i) \cap B_R) \right\|_1 \right) \\ &\quad \le 2\varepsilon \qquad \text{[by (35) and (28)]}. \end{aligned} \tag{42}$$

For each $i \in I$, let us now consider the partition $\hat{\mathbf{G}}^i$ obtained applying Theorem 3.4 to $\mathbf{G}^i$ on the balls $B_\rho(\mathbf{y}_k^i)$ so that, by (30),

$$\hat{\mathbf{G}}^i \triangle \mathbf{G}^i \Subset B_{R-1} \cap \Pi^i \subseteq B_{R-\alpha} \cap \Pi^i \subseteq (\Omega_n^i - \mathbf{x}_n^i) + \alpha \nu^i, \tag{43}$$

$$\left| (\hat{G}_j^i - \alpha \nu^i + \mathbf{x}_n^i) \cap \Omega_n^i \right| = |(E_n)_j \cap \Omega_n| \qquad \forall i \in I \quad \forall j \in J. \tag{44}$$

This is achieved by means of variations which changes the volume of each region G_j^i, with $j \in J$, by a (possibly negative) amount a_j^i with, using (27) and (35),

$$\begin{aligned} \left| a_j^i \right| &\le \left| (G_j^i - \alpha \nu^i) \triangle G_j^i \cap B_R \right| + \left| G_j^i \triangle F_j^i \cap B_R \right| + \left| F_j^i \triangle ((E_n)_j - \mathbf{x}_n^i) \cap B_R \right| \\ &\le \frac{1}{2} \left(\frac{\varepsilon}{N^2 C_1} \right)^{\frac{d}{d-1}} + 0 + \frac{1}{2} \left(\frac{\varepsilon}{N^2 C_1} \right)^{\frac{d}{d-1}} = \left(\frac{\varepsilon}{N^2 C_1} \right)^{\frac{d}{d-1}} \end{aligned}$$

for $j \in J$, and $a_k^i = 0$ for $k \notin J$ (see also [17, Theorem 2.13] where this step is explained in detail). Using also (27), we have

$$\operatorname{Per}(\hat{\mathbf{G}}^i, B_{R-\rho}) - \operatorname{Per}(\mathbf{G}^i, B_{R-\rho}) \le N C_1 \sum_{j=1}^N \left| a_j^i \right|^{\frac{d-1}{d}} \le \varepsilon. \tag{45}$$

Define

$$\hat{\mathbf{E}}_n := [\mathbf{E}_n \setminus \Omega_n] \cup \bigcup_{i \in I} \left[(\hat{\mathbf{G}}^i - \alpha \nu^i + \mathbf{x}_n^i) \cap \Omega_n^i \right]$$

so that $\hat{\mathbf{E}}_n$ is a competitor to the constrained minimality of $\mathbf{E}_n$ in $\bar{B}_{R_n}$

$$\operatorname{Per}(\mathbf{E}_n, \bar{B}_{R_n}) \le \operatorname{Per}(\hat{\mathbf{E}}_n, \bar{B}_{R_n})$$

and since $\hat{\mathbf{E}}_n \triangle \mathbf{E}_n \subseteq \bar{B}_{R_n} \cap \bigcup_{i \in I} \bar{B}_{R-\rho}(\mathbf{x}_n^i)$ we have

$$\sum_{i \in I} \operatorname{Per}(\mathbf{E}_n, \bar{B}_{R-\rho}(\mathbf{x}_n^i)) \le \sum_{i \in I} \operatorname{Per}(\hat{\mathbf{E}}_n, \bar{B}_{R-\rho}(\mathbf{x}_n^i)). \tag{46}$$

Let

$$\mathbf{H}^i := (\mathbf{S} - \mathbf{x}_n^i) \setminus B_{R_n} \cup (\mathbf{T}^i \cap B_{R_n}).$$

We need to estimate $\operatorname{Per}(\mathbf{H}^i, B_R(\mathbf{x}_n^i) \cap \partial B_{R_n})$. We assume $\left| \mathbf{x}_n^i \right| \to +\infty$ otherwise there is nothing to prove.

By (40), if we take $\delta = \frac{\varepsilon}{4\theta R^{d-2}}$ we have

$$(\mathbf{T}^i \triangle (\mathbf{S} - \mathbf{x}_n^i)) \cap B_R \subseteq (\partial \mathbf{T}^i)_\delta$$

and hence

$$\begin{aligned} &\mathrm{Per}(\mathbf{H}^i, B_R \cap \partial B_{R_n}(-\mathbf{x}_n^i)) \\ &\quad = \frac{1}{2}\sum_{k=1}^{N} \mathcal{H}^{d-1}(((S_k - \mathbf{x}_n^i)\triangle T_k^i) \cap B_R \cap \partial B_{R_n}(-\mathbf{x}_n^i)) \\ &\quad \le \mathcal{H}^{d-1}((\partial \mathbf{T}^i)_\delta \cap B_R \cap \partial B_{R_n}(-\mathbf{x}_n^i)) \\ &\quad \le 2\mathcal{H}^{d-1}((\partial \mathbf{T}_0^i)_\delta \cap B_R \cap \partial \Pi^i) \qquad \text{[by (25)]} \\ &\quad \le 4\theta R^{d-2}\delta \le \varepsilon \qquad \text{[by (24)]}. \end{aligned} \tag{47}$$

We have

$$\begin{aligned} &\mathrm{Per}(\tilde{\mathbf{E}}_n, \bar{B}_{R-\rho}(\mathbf{x}_n^i) \setminus B_{R_n}) \\ &\quad = \mathrm{Per}(\tilde{\mathbf{E}}_n - \mathbf{x}_n^i, \bar{B}_{R-\rho} \setminus B_{R_n}(-\mathbf{x}_n^i)) \\ &\quad = \mathrm{Per}(\mathbf{H}^i, \bar{B}_{R-\rho} \setminus B_{R_n}(-\mathbf{x}_n^i)) \\ &\quad = \mathrm{Per}(\mathbf{S} - \mathbf{x}_n^i, \bar{B}_{R-\rho} \setminus \bar{B}_{R_n}(-\mathbf{x}_n^i)) \\ &\qquad + \mathrm{Per}(\mathbf{H}^i, B_{R-\rho} \cap \partial B_{R_n}(-\mathbf{x}_n^i)) \\ &\quad \le \mathrm{Per}(\mathbf{S} - \mathbf{x}_n^i, B_R \setminus (\bar{\Pi}^i - \alpha\nu^i)) + \varepsilon \qquad \text{[by (36) and (47)]} \\ &\quad \le \mathrm{Per}(\mathbf{T}^i, B_R \setminus (\bar{\Pi}^i - \alpha\nu^i)) + 2\varepsilon \qquad \text{[by (39)]} \\ &\quad \le \mathrm{Per}(\mathbf{T}^i, B_R \setminus (\Pi^i + 2\alpha\nu^i)) + 3\varepsilon \qquad \text{[by (31)]} \\ &\quad \le \mathrm{Per}(\mathbf{T}^i, B_{R-\rho} \setminus (\Pi^i + 2\alpha\nu^i)) + 4\varepsilon \qquad \text{[by (32)]} \\ &\quad = \mathrm{Per}(\mathbf{G}^i, \bar{B}_{R-\rho} \setminus (\Pi^i + 2\alpha\nu^i)) + 4\varepsilon \end{aligned}$$

while

$$\begin{aligned} \mathrm{Per}(\tilde{\mathbf{E}}_n, \bar{B}_{R-\rho}(\mathbf{x}_n^i) \cap B_{R_n}) &= \mathrm{Per}(\mathbf{G}^i - \alpha\nu^i, B_{R-\rho} \cap B_{R_n}(-\mathbf{x}_n^i)) \\ &\quad + \mathrm{Per}(\tilde{\mathbf{E}}_n, \partial B_{R-\rho}(\mathbf{x}_n^i) \cap B_{R_n}) \\ &\le \mathrm{Per}(\mathbf{G}^i, (B_{R-\rho} \cap B_{R_n}(-\mathbf{x}_n^i)) + \alpha\nu^i) + 2\varepsilon \quad \text{[by (42)]} \\ &\le \mathrm{Per}(\mathbf{G}^i, B_{R+\alpha} \cap (\Pi^i + 2\alpha\nu^i)) + 2\varepsilon \qquad \text{[by (37)]} \\ &\le \mathrm{Per}(\mathbf{G}^i, B_{R-\alpha} \cap (\Pi^i + 2\alpha\nu^i)) + 3\varepsilon \qquad \text{[by (29)]} \end{aligned}$$

so, recalling that $(\hat{\mathbf{E}}_n \triangle \tilde{\mathbf{E}}_n) \cap B_R(\mathbf{x}_n^i) \subseteq \mathbf{x}_n^i - \alpha\nu^i + (\hat{\mathbf{G}}^i \triangle \mathbf{G}^i) \Subset B_{R-1}(\mathbf{x}_n^i) - \alpha\nu^i \subseteq B_{R-1+\alpha}(\mathbf{x}_n^i) \subseteq B_{R-\alpha}(\mathbf{x}_n^i) \Subset B_{R-\rho}$

$$\begin{aligned} \mathrm{Per}(\hat{\mathbf{E}}_n, \bar{B}_{R-\rho}(\mathbf{x}_n^i)) &\le \mathrm{Per}(\tilde{\mathbf{E}}_n, \bar{B}_{R-\rho}(\mathbf{x}_n^i)) + \varepsilon \qquad \text{[by (45)]} \\ &= \mathrm{Per}(\tilde{\mathbf{E}}_n, \bar{B}_{R-\rho}(\mathbf{x}_n^i) \setminus B_{R_n}) \\ &\quad + \mathrm{Per}(\tilde{\mathbf{E}}_n, \bar{B}_{R-\rho}(\mathbf{x}_n^i) \cap B_{R_n}) + \varepsilon \\ &\le \mathrm{Per}(\mathbf{G}^i, \bar{B}_{R-\rho}) + 8\varepsilon \end{aligned} \tag{48}$$

and the proof is concluded:

$$\begin{aligned}
\sum_{i\in I}\operatorname{Per}(\mathbf{F}^i, B_R) &\le \sum_{i\in I}\left[\operatorname{Per}(\mathbf{E}_n, B_R(\mathbf{x}_n^i)) + \varepsilon\right] && \text{[by (38)]}\\
&\le \sum_{i\in I}\left[\operatorname{Per}(\mathbf{E}_n, \bar{B}_{R-\rho}(\mathbf{x}_n^i)) + 2\varepsilon\right] && \text{[by (33)]}\\
&\le \sum_{i\in I}\left[\operatorname{Per}(\hat{\mathbf{E}}_n, \bar{B}_{R-\rho}(\mathbf{x}_n^i)) + 2\varepsilon\right] && \text{[by (46)]}\\
&\le \sum_{i\in I}\left[\operatorname{Per}(\mathbf{G}^i, \bar{B}_{R-\rho}) + 10\varepsilon\right] && \text{[by (48)]}\\
&\le \sum_{i\in I}\left[\operatorname{Per}(\mathbf{G}^i, B_R) + 10\varepsilon\right]
\end{aligned}$$

recalling that $\mathbf{F}^i \triangle \mathbf{G}^i \subseteq K \Subset B_{R-1}$, and that I is finite. □

4. Monotonicity formula and blow-down partitions

The following result is proven in [12, Theorem 1.9, 2.1 and Proposition 4.1], along the lines of Allard [2].

Theorem 4.1 (Asymptotic expansion of planelike stationary varifolds)**.** *Let $E \subseteq \mathbb{R}^d \setminus \bar{B}_1$ be a set of locally finite perimeter such that $\mathbf{v}(\partial^* E \setminus \bar{B}_1, 1)$ is a stationary varifold in $\mathbb{R}^d \setminus \bar{B}_1$. Suppose that*

$$\lim_{\rho\to+\infty} \frac{\operatorname{Per}(E, B_\rho \setminus \bar{B}_1)}{\omega_{d-1}\rho^{d-1}} = 1,$$

and, for some $r_k \to +\infty$,

$$\frac{E}{r_k} \to \left\{x \in \mathbb{R}^d : x_d < 0\right\} \qquad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^d) \text{ as } k \to +\infty.$$

Then there exists $f\colon \mathbb{R}^{d-1} \setminus \bar{B}_1 \to \mathbb{R}$ such that $E = \left\{(x,y) \in \mathbb{R}^{d-1} \times \mathbb{R} : f(x) < y\right\}$, up to a negligible set. Moreover, if $d \ge 4$ the function f has the asymptotic expansion

$$f(x) = a + \frac{b}{|x|^{d-3}} + O\left(\frac{1}{|x|^{d-2}}\right) \qquad \text{as } |x| \to +\infty, \tag{49}$$

with $a, b \in \mathbb{R}$.

In particular $\frac{E}{r}$ converges to a half-space as $r \to +\infty$, and the blow-down is unique.

We also recall a result from [7, Corollary 4.6].

Theorem 4.2 (Existence of blow-down partitions)**.** *Let $\mathbf{F}$ be a N-isoperimetric partition of $\mathbb{R}^d$, and let $R_k \to +\infty$. Then, up to a non-relabeled subsequence, as $k \to \infty$ we have*

$$\begin{aligned}
&\frac{\mathbf{F}}{R_k} \to \mathbf{F}_\infty \qquad \text{in } L^1_{loc}(\mathbb{R}^d),\\
&\frac{\operatorname{Per}(\mathbf{F}, B_{R_k})}{R_k^{d-1}} \to \operatorname{Per}(\mathbf{F}_\infty, B_1),
\end{aligned}$$

where $\mathbf{F}_\infty$ is a (possibly improper) N-isoperimetric conical partition of $\mathbb{R}^d$.

We plan to extend the previous results to isoperimetric partitions in a half-space Π of $\mathbb{R}^d$, with boundary given by a $d-2$-dimensional affine space contained in $\partial\Pi$. We start by showing the following monotonicity property.

Theorem 4.3 (Monotonicity formula for isoperimetric partitions in a cone)**.** *Let Π be an open cone in $\mathbb{R}^d$ with vertex in 0. Let $\mathbf{F}$ be a Caccioppoli N-partition of $\mathbb{R}^d$ such that each region of $\mathbf{F}\setminus\bar{\Pi}$ is also a cone with vertex in 0 (boundary condition).*

Suppose that $\mathbf{F}$ is locally isoperimetric in $\bar{\Pi}$. Suppose there exists $R>0$ be such that $F^k\subseteq B_R$ if $\left|F^k\right|<+\infty$. Then

$$\rho\mapsto\frac{\operatorname{Per}(\mathbf{F},B_\rho\setminus B_R)}{\rho^{d-1}}$$

is non-decreasing on $[R,+\infty)$, so that

$$\rho\mapsto\frac{\operatorname{Per}(\mathbf{F},B_\rho\cap\bar{\Pi})}{\rho^{d-1}}=\frac{\operatorname{Per}(\mathbf{F},B_\rho\setminus B_R)+\operatorname{Per}(\mathbf{F},B_R)}{\rho^{d-1}}-\operatorname{Per}(\mathbf{F},B_1\setminus\bar{\Pi})$$

has limit as $\rho\to+\infty$.

Proof. Without loss of generality suppose that $\left|F^k\right|<+\infty$ if and only if $k\leq M$ for some $M<N$. Let $u(\rho):=\operatorname{Per}(\mathbf{F},B_\rho)$. By the coarea inequality we know that for a.e. $\rho>0$ one has

$$u'(\rho)\geq\mathcal{H}^{d-2}(\partial\mathbf{F}\cap\partial B_\rho).$$

For any $\rho>R$ lets consider the cone with the same boundary of $\mathbf{F}$ on ∂B_ρ which is a partition $\mathbf{C}_\rho=(C^1_\rho,\dots,C^N_\rho)$ such that $x\in C^k_\rho$ if and only if $\rho\frac{x}{|x|}\in F_k$. Notice that $C^k_\rho=\emptyset$ for $k\leq M$. For a.e. $\rho>R$ one has

$$\operatorname{Per}(\mathbf{C}_\rho,\bar{B}_\rho)=\frac{\rho}{d-1}\mathcal{H}^{d-2}(\partial\mathbf{F}\cap\partial B_\rho)$$

and $\mathbf{C}_\rho\triangle\mathbf{F}\subseteq\bar{\Pi}$. For $\rho>R$ we are going to build a competitor to $\mathbf{F}$ by taking $\mathbf{F}_\rho=(F^1_\rho,\dots,F^N_\rho)$ with

$$F^k_\rho:=\begin{cases}F^k & \text{if } 1\leq k\leq M\\ (F^k\setminus B_\rho)\cup(C^k_\rho\cap B_\rho)\setminus\displaystyle\bigcup_{j=1}^M F_j & \text{if } M<k\leq N\end{cases}$$

so that $\mathbf{F}_\rho\triangle\mathbf{F}\subseteq\bar{\Pi}$ and, for a.e. $\rho>R$,

$$\operatorname{Per}(\mathbf{F}_\rho,B_\rho)\leq\frac{r}{d-1}\mathcal{H}^{d-2}(\partial\mathbf{F}\cap\partial B_\rho)+\operatorname{Per}(\mathbf{F},B_R).$$

Hence for a.e. $\rho>R$

$$\begin{aligned}u(\rho)=P(\mathbf{F},B_\rho)\leq P(\mathbf{F}_\rho,B_\rho)&\leq\frac{\rho}{d-1}\mathcal{H}^{d-2}(\partial\mathbf{F}\cap\partial B_\rho)+\operatorname{Per}(\mathbf{F},B_R)\\ &\leq\frac{\rho}{d-1}u'(\rho)+\operatorname{Per}(\mathbf{F},B_R).\end{aligned}$$

This is equivalent to

$$u'(\rho)\cdot\rho^{1-d}-(d-1)\cdot u(\rho)\cdot\rho^{-d}+(d-1)\cdot\operatorname{Per}(\mathbf{F},B_R)\cdot\rho^{-d}\geq 0$$

or

$$\frac{d}{d\rho}\left[u(\rho)\cdot\rho^{1-d}-\operatorname{Per}(\mathbf{F},B_R)\cdot\rho^{1-d}\right]\geq 0$$

which means that
$$\rho \mapsto \frac{\mathrm{Per}(\mathbf{F}, B_\rho) - \mathrm{Per}(\mathbf{F}, B_R)}{\rho^{d-1}}$$
is non-decreasing. Noting that $\mathbf{F}$ is a cone outside $\bar{\Pi}$, the function
$$\rho \mapsto \frac{\mathrm{Per}(\mathbf{F}, B_\rho \setminus \bar{\Pi})}{\rho^{d-1}}$$
is constant, and the statement follows. $\square$

The following result readily follows from Theorem 4.3. The proofs remain the same as in [7, Corollary 4.6].

Corollary 4.4 (Existence of blow-down partitions in Π)**.** *Let* $\mathbf{F}$ *and* Π *be as in Theorem 4.3, then there exists, and it is finite, the limit*
$$\Theta^{\Pi}_{\infty}(\mathbf{F}) := \lim_{r\to+\infty} \frac{\mathrm{Per}(\mathbf{F}, B_r \cap \Pi)}{r^{d-1}}.$$
Moreover, let $r_k \to +\infty$*. Then, up to a non-relabeled subsequence, as* $k \to \infty$ *we have*
$$\frac{\mathbf{F}}{r_k} \to \mathbf{F}_\infty \qquad \text{in } L^1_{loc}(\mathbb{R}^d),$$
$$\mathrm{Per}(\mathbf{F}_\infty, B_1 \cap \bar{\Pi}) = \Theta^{\Pi}_{\infty}(\mathbf{F})$$
where $\mathbf{F}_\infty$ *is a conical partition, locally minimal in* $\bar{\Pi}$*.*

5. Existence of non-standard isoperimetric partitions

The following definition provides a way to measure the perimeter that can be gained by removing a large portion of a partition $\mathbf{E}$ and replacing it with its blow-down partition.

Definition 5.1 (defect)**.** Let $\mathbf{E}$ be a Caccioppoli 3-partition of $\mathbb{R}^d$ with $|E_1| < +\infty$ and suppose that $\mathbf{E}$ has a unique blow-down $\mathbf{E}_\infty$. Let $\Pi \subseteq \mathbb{R}^d$ be a positive cone with vertex in $0 \in \mathbb{R}^d$. We define the *defect* of $\mathbf{E}$ in $\bar{\Pi}$ as

(50)
$$\Delta^{\bar{\Pi}}_{\mathbf{E}} := \liminf_{r\to+\infty} \frac{\mathrm{Per}(\mathbf{E}, \bar{\Pi} \cap B_r) - \mathrm{Per}(\mathbf{E}_\infty, \bar{\Pi} \cap B_r)}{|E_1|^{\frac{d-1}{d}}}.$$

In the case when $\Pi = \mathbb{R}^d$ we simply write:
$$\Delta_{\mathbf{E}} := \Delta^{\mathbb{R}^d}_{\mathbf{E}}.$$

Remark 5.2. Notice that, if $\mathbf{E} = (E, \emptyset, \mathbb{R}^d \setminus E)$ is the 3-partition associated to a set E with $|E| < +\infty$, then
$$\Delta_{\mathbf{E}} = \frac{\mathrm{Per}(E)}{|E|^{\frac{d-1}{d}}}$$
is the usual *isoperimetric ratio* of E.

Definition 5.3 (lens partition)**.** We say that a 3-partition $\mathbf{L} = (L^1, L^2, L^3)$ of $\Omega \subseteq \mathbb{R}^d$ is a lens partition in Ω if $L^1 \subseteq \bar{\Omega}$ is a lens:
$$L^1 = B_R(p) \cap B_R(q), \qquad \text{with } p, q \in \mathbb{R}^d,\ |p-q| = R$$
and $L^2 = H \setminus L^1$, $L^3 = \mathbb{R}^d \setminus (H \cup L^1)$ where ∂H is the hyperplane equidistant from the points p, q:
$$H = \left\{x \in \mathbb{R}^d\colon\ |x-p| \le |x-q|\right\}.$$

Definition 5.4 (Simons' cone, barrel)**.** We call *Simons' cone partition* the 3-partition $\mathbf{S} = (S_1, S_2, S_3)$ of $\mathbb{R}^8$ defined by

$$\begin{aligned}
S_1 &:= \emptyset, \\
S_2 &:= \left\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^4 \times \mathbb{R}^4 \colon\ |\mathbf{x}| \le |\mathbf{y}|\right\}, \\
S_3 &:= \mathbb{R}^d \setminus S_2.
\end{aligned}$$

We call *barrel partition* the 3-partition $\mathbf{Q} = (Q_1, Q_2, Q_3)$ of $\mathbb{R}^8$ defined by

$$\begin{aligned}
Q_1 &:= \mathbb{B}^4 \times \mathbb{B}^4 = \left\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^4 \times \mathbb{R}^4 \colon\ |\mathbf{x}| \le 1, |\mathbf{y}| \le 1\right\}, \\
Q_2 &:= S_2 \setminus Q_1, \\
Q_3 &:= S_3 \setminus Q_1.
\end{aligned}$$

Remark 5.5. Clearly $\mathbf{m}(\mathbf{S}) = (0, +\infty, +\infty)$ and it is well known that $\mathbf{S}$ is an isoperimetric partition since S_2 is a locally minimal set as proved in [4] (see also [8]).

Lemma 5.6 (defects of lens and barrel partitions)**.** *Let* $\mathbf{L}$ *be a lens partition, then*

$$\Delta_{\mathbf{L}} = 4\sqrt[8]{4\pi^3\left[\frac{16}{9} - \frac{93}{35}\sqrt{3}\right]} \approx 7.29. \tag{51}$$

Let $\mathbf{Q}$ *be the barrel partition, then*

$$\Delta_{\mathbf{Q}} = \sqrt[4]{8}\sqrt{\pi}\left[4 - \frac{8}{7}\sqrt{2}\right] \approx 7.10.$$

Proof. We have

$$\begin{aligned}
\omega_d := \left|\mathbb{B}^d\right| &= \frac{\pi^{\frac{d}{2}}}{\frac{d}{2}!} = \begin{cases} \frac{\pi^k}{k!} & \text{if } d = 2k \text{ is even,} \\ \frac{\pi^k 2^k}{d!!} & \text{if } d = 2k+1 \text{ is odd,} \end{cases} \\
\mathcal{H}^d(\mathbb{S}^d) &= (d+1)\omega_{d+1}
\end{aligned}$$

whence

$$\left|\mathbb{B}^4\right| = \frac{\pi^2}{2}, \qquad \mathcal{H}^3(\mathbb{S}^3) = 2\pi^2, \qquad \left|\mathbb{B}^7\right| = \frac{16}{105}\pi^3, \qquad \mathcal{H}^6(\mathbb{S}^6) = 7\omega_7 = \frac{16}{15}\pi^3.$$

Let $\mathbf{S} = (\emptyset, S_2, \mathbb{R}^8 \setminus S_2)$ be the Simons' cone partition and let $\mathbf{Q} = (Q_1, S_2 \setminus Q_1, \mathbb{R}^8 \setminus (S_1 \cup Q_1))$ be the *barrel* partition (see Definition 5.4). Then

$$\begin{aligned}
|Q_1| &= \left|\mathbb{B}^4\right|^2 = \omega_4^2 = \frac{\pi^4}{4}, \\
\mathrm{Per}(Q_1) &= 2\mathcal{H}^3(\mathbb{S}^3)\mathcal{H}^4(\mathbb{B}^4) = 2\pi^4, \\
\mathrm{Per}(S_2, Q_1) &= \mathcal{H}^3(\mathbb{S}^3)^2 \int_0^1 x^3 x^3 \sqrt{2}dx = \frac{4}{7}\sqrt{2}\pi^4.
\end{aligned}$$

Now consider the lens partition $\mathbf{L} = (L_1, H \setminus L_1, \mathbb{R}^8 \setminus (H \cup L_1)$ with

$$\begin{aligned}
H &= \left\{x \in \mathbb{R}^d \colon x_8 \ge 0\right\} \\
L_1 &= (\mathbb{B}^8 + \frac{e_8}{2}) \cap (\mathbb{B}^8 - \frac{e_8}{2})
\end{aligned}$$

where $e_8 = (0, \dots, 0, 1) \in \mathbb{R}^8$ (see Definition 5.3).

One has

$$
\begin{aligned}
|L_1| &= 2\int_{\frac{1}{2}}^{1} \left|\sqrt{1-y^2}\cdot\mathbb{B}^7\right| dy = 2\omega_7 \int_{\frac{1}{2}}^{1} (1-y^2)^{\frac{7}{2}}\, dy \\
&= 2\omega_7 \int_0^{\frac{\pi}{3}} \sin^8\theta\, d\theta = \omega_7\left(\frac{35}{192}\pi - \frac{279}{1024}\sqrt{3}\right) \\
&= \frac{\pi^4}{36} - \frac{93}{2240}\sqrt{3}\pi^3, \\
\operatorname{Per}(H, L_1) &= \left|\frac{\sqrt{3}}{2}\mathbb{B}^7\right| = \frac{27}{128}\sqrt{3}\omega_7 = \frac{9}{280}\sqrt{3}\pi^3.
\end{aligned}
$$

If we let $V(r) = |L_1 \cap H|$, on one hand we have

$$
V'(1) = \frac{1}{2}\mathcal{H}^7(\partial L_1) - \frac{1}{2}\mathcal{H}^7\left(\frac{\sqrt{3}}{2}\mathbb{B}^7\right)
$$

and on the other hand we have $2V(r) = r^8\,|L_1|$ hence

$$
\operatorname{Per}(L_1) = 8\,|L_1| + \omega_7\left(\frac{\sqrt{3}}{2}\right)^7 = 8\,|L_1| + \frac{16}{105}\cdot\frac{27}{128}\sqrt{3}\pi^3 = \frac{2}{9}\pi^4 - \frac{3}{10}\sqrt{3}\pi^3
$$

In view of Remark 5.2 we can finally compute and compare the defects:

$$
\begin{aligned}
\Delta_{\mathbf{Q}} &= \frac{\operatorname{Per}(Q_1) - \operatorname{Per}(S_2, Q_1)}{|Q_1|^{\frac{7}{8}}} = \frac{2\pi^4 - \frac{4}{7}\sqrt{2}\pi^4}{\left(\frac{\pi^4}{4}\right)^{\frac{7}{8}}} \\
&= \sqrt[4]{8}\sqrt{\pi}\left[4 - \frac{8}{7}\sqrt{2}\right] \approx 7.10, \\
\Delta_{\mathbf{L}} &= \frac{\operatorname{Per}(L_1) - \operatorname{Per}(H, L_1)}{|L_1|^{\frac{7}{8}}} = \frac{\frac{2}{9}\pi^4 - \frac{93}{280}\sqrt{3}\pi^3}{\left(\frac{\pi^4}{36} - \frac{93}{2240}\sqrt{3}\pi^3\right)^{\frac{7}{8}}} \\
&= 4\sqrt[8]{4\pi^3\left[\frac{16}{9}\pi - \frac{93}{35}\sqrt{3}\right]} \approx 7.29.
\end{aligned}
$$

□

Lemma 5.7. *Let $R_n > 0$, $R_n \to +\infty$ be given. Let $\mathbf{S}$ be the Simons' partition (see Definition 5.4). Let $\mathbf{E}_n = ((E_n)_1, (E_n)_2, (E_n)_3)$ be a 3-partition of $\mathbb{R}^8$ such that $\mathbf{E}_n \setminus \bar{B}_{R_n} = \mathbf{S} \setminus \bar{B}_{R_n}$, and $|(E_n)_1| = 1$. Suppose that for all $i \in \mathbb{N}$, $i > 0$, there exist $\mathbf{x}_n^i \in \mathbb{R}^d$, and $\mathbf{F}^i$, such that*

$$
\begin{gathered}
\mathbf{E}_n - \mathbf{x}_n^i \xrightarrow{L^1_{\mathrm{loc}}} \mathbf{F}^i, \\
\sum_{i=1}^{+\infty} \left|F_1^i\right| = 1, \\
\lim_{n\to+\infty} \left|\mathbf{x}_n^i - \mathbf{x}_n^j\right| = +\infty \qquad \text{for all } i \neq j.
\end{gathered}
$$

Let Π^i be the L^1_{loc}-limit of the balls $B_{R_n} - \mathbf{x}_n^i$ as $n \to \infty$. Clearly $\bar{\Pi}^i$ is either $\mathbb{R}^8$ (if $R_n - \left|\mathbf{x}_n^i\right| \to \infty$) or a closed half-space. Suppose moreover that there exist partitions

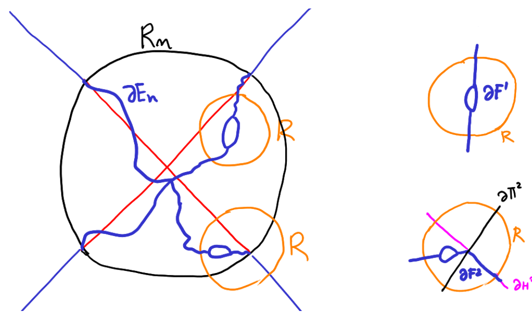

FIGURE 3. A reference picture for the proof of Lemma 5.7.

$\mathbf{F}^i_\infty = (\emptyset, (F^i_\infty)_2, (F^i_\infty)_3)$, $\mathbf{T}^i = (\emptyset, T^i_2, T^i_3)$, $\mathbf{H}^i = (\emptyset, H^i_2, H^i_3)$, with $\partial \mathbf{H}^i = \emptyset$ or $\partial \mathbf{H}^i$ is a hyperplane passing through the origin, so that

$$\frac{\mathbf{F}^i}{r} \xrightarrow{L^1_{\mathrm{loc}}} \mathbf{F}^i_\infty \qquad \text{as } r \to +\infty \tag{52}$$

$$\mathbf{S} - \mathbf{x}^i_n \xrightarrow{L^1_{\mathrm{loc}}} \mathbf{T}^i \quad \text{as } n \to +\infty \tag{53}$$

$$(\mathbf{F}^i_\infty + \mathbf{y}^i) \cap \Pi^i = (\mathbf{H}^i + \mathbf{y}^i) \cap \Pi^i \tag{54}$$

$$(\mathbf{F}^i_\infty + \mathbf{y}^i) \setminus \Pi^i = \mathbf{T}^i \setminus \Pi^i, \tag{55}$$

for some $\mathbf{y}^i \in \mathbb{R}^d$. Suppose also

$$\left\| \mathbf{m}((\mathbf{F}^i \triangle (\mathbf{H}^i + \mathbf{y}^i)) \cap (B_r \setminus B_{r-1}) \cap \Pi^i) \right\|_1 \to 0. \tag{56}$$

Then

$$\limsup_{n \to +\infty} \Delta_{\mathbf{E}_n} \geq \inf_i \Delta^{\bar{\Pi}^i}_{\mathbf{F}^i}.$$

Proof. Fix $\varepsilon \in (0,1)$. Take N large enough such that

$$\sum_{i=N+1}^{+\infty} \left| F^i_1 \right| \leq \sum_{i=N+1}^{+\infty} \left| F^i_1 \right|^{\frac{7}{8}} < \varepsilon. \tag{57}$$

Take also $r > 1$ large enough so that such that $\sum_{i=1}^N \left| F^i_1 \cap B_{r-1} \right| > 1 - 2\varepsilon$ and

$$\sum_{i=1}^N \left\| \mathbf{m}((\mathbf{F}^i \triangle (\mathbf{H}^i + \mathbf{y}^i)) \cap (B_r \setminus B_{r-1}) \cap \Pi^i) \right\|_1 < \varepsilon,$$

$$\sum_{i=1}^N \left(\mathrm{Per}(\mathbf{F}^i, B_\rho \cap \bar{\Pi}^i) - \mathrm{Per}(\mathbf{H}^i + \mathbf{y}^i, B_\rho \cap \bar{\Pi}^i) \right) \geq \left(\sum_{i=1}^N \Delta^{\bar{\Pi}^i}_{\mathbf{F}^i} \cdot \left| F^i_1 \right|^{\frac{7}{8}} \right) - \varepsilon, \tag{58}$$

for all $\rho \in [r-1, r]$.

Fix n large enough so that the balls $B_r(\mathbf{x}_n^i)$ are pairwise disjoint, and

$$\sum_{i=1}^N \left\|\mathbf{m}(((\mathbf{E}_n - \mathbf{x}_n^i)\triangle\mathbf{F}^i)\cap B_r)\right\|_1 < \varepsilon,$$

$$\sum_{i=1}^N \left|((B_{R_n} - \mathbf{x}_n^i)\triangle\Pi^i)\cap B_r\right| < \varepsilon$$

$$\operatorname{Per}(\mathbf{H}^i + \mathbf{y}^i, B_r \cap \bar{B}_{R_n}(-\mathbf{x}_n^i)\setminus\bar{\Pi}^i) < \frac{\varepsilon}{N},$$

so that

$$\sum_{i=1}^N \left\|\mathbf{m}((\mathbf{E}_n - \mathbf{x}_n^i)\triangle(\mathbf{H}^i+\mathbf{y}^i))\cap(B_r\setminus B_{r-1})\cap(B_{R_n}-\mathbf{x}_n^i)\right\|_1 < 3\varepsilon. \tag{59}$$

As a consequence, for $\rho\in[r-1,r]$

$$\begin{aligned}\operatorname{Per}(\mathbf{H}^i+\mathbf{y}^i, B_\rho\cap\bar{B}_{R_n}(-\mathbf{x}_n^i)) &\le \operatorname{Per}(\mathbf{H}^i+\mathbf{y}^i, B_\rho\cap\bar{\Pi}^i) \\ &\quad + \operatorname{Per}(\mathbf{H}^i+\mathbf{y}^i, B_\rho\cap\bar{B}_{R_n}(-\mathbf{x}_n^i)\setminus\bar{\Pi}^i) \qquad (60)\\ &\le \operatorname{Per}(\mathbf{H}^i+\mathbf{y}^i, B_\rho\cap\bar{\Pi}^i) + \frac{\varepsilon}{N}. \qquad (61)\end{aligned}$$

By the semicontinuity of perimeter and the convergence of $B_{R_n}(-\mathbf{x}_n^i)$ to Π^i, we can also suppose that

$$\operatorname{Per}(\mathbf{F}^i, B_r\cap\bar{\Pi}^i) \le \operatorname{Per}(\mathbf{E}_n, B_r(\mathbf{x}_n^i)\cap\bar{B}_{R_n}) + \frac{\varepsilon}{N}, \tag{62}$$

indeed, for $\delta>0$ we have

$$\begin{aligned}\operatorname{Per}(\mathbf{F}^i, B_r\cap\bar{\Pi}^i) &\le \liminf_n \operatorname{Per}(\mathbf{E}_n - \mathbf{x}_n^i, B_r\cap B_{R_n+\delta}(-\mathbf{x}_n^i)) \\ &= \liminf_n \big[\operatorname{Per}(\mathbf{E}_n, B_r(\mathbf{x}_n^i)\cap\bar{B}_{R_n}) \\ &\quad + \operatorname{Per}(\mathbf{S}, B_r(\mathbf{x}_n^i)\cap(B_{R_n+\delta}\setminus\bar{B}_{R_n}))\big] \\ &\le \liminf_n \operatorname{Per}(\mathbf{E}_n, B_r(\mathbf{x}_n^i)\cap\bar{B}_{R_n}) + C\delta r^{d-1},\end{aligned}$$

which proves (62) by choosing δ such that $C\delta r^{d-1} < \varepsilon/(2N)$ and n large enough.

Let now

$$\mathbf{G}_n^i := ((\mathbf{x}_n^i + \mathbf{y}^i + \mathbf{H}^i)\cap B_{R_n})\cup(\mathbf{S}\setminus B_{R_n}). \tag{63}$$

Notice that

$$\mathbf{G}_n^i - \mathbf{x}_n^i \to ((\mathbf{y}^i+\mathbf{H}^i)\cap\Pi^i)\cup(\mathbf{T}^i\setminus\Pi^i) = \mathbf{y}^i + \mathbf{F}_\infty^i,$$

where the limit is locally in the Hausdorff distance. Moreover, we have

$$\limsup_n \operatorname{Per}(\mathbf{G}_n^i - \mathbf{x}^i, B_\rho\cap\bar{B}_{R_n}(-\mathbf{x}^i)) \le \operatorname{Per}(\mathbf{H}^i+\mathbf{y}^i, B_\rho\cap\bar{\Pi}^i). \tag{64}$$

Indeed, recalling from (61) for $\delta>0$ we have

$$\begin{aligned}\limsup_n \operatorname{Per}(\mathbf{G}_n^i - \mathbf{x}_n^i, B_\rho\cap\bar{B}_{R_n}(-\mathbf{x}_n^i)) &\le \limsup_n \operatorname{Per}(\mathbf{G}_n^i - \mathbf{x}^i, B_\rho\cap B_{R_n+\delta}(-\mathbf{x}_n^i)) \\ &\le \operatorname{Per}(\mathbf{H}^i+\mathbf{y}^i, B_\rho\cap(\Pi^i)_{2\delta})) \\ &\quad + \operatorname{Per}(\mathbf{T}^i, B_\rho\cap((\Pi^i)_{2\delta})\setminus\bar{\Pi}^i)),\end{aligned}$$

which gives (64) letting $\delta \to 0$. Hence, from (64), for n large enough it holds good

$$\sum_{i=1}^{N} \operatorname{Per}(\mathbf{G}_n^i - \mathbf{x}^i, B_\rho \cap \bar{B}_{R_n}(-\mathbf{x}^i)) \leq \sum_{i=1}^{N} \operatorname{Per}(\mathbf{H}^i + \mathbf{y}^i, B_\rho \cap \bar{\Pi}^i) + \varepsilon. \tag{65}$$

Let

$$\hat{\mathbf{E}}_n = (\emptyset, (E_n)_1 \cup (E_n)_2, (E_n)_3).$$

Clearly for all Borel $B \subseteq \mathbb{R}^d$ one has

$$\operatorname{Per}(\hat{\mathbf{E}}_n, B) \leq \operatorname{Per}(\mathbf{E}_n, B). \tag{66}$$

By construction we have

$$\left\| \mathbf{m}(\hat{\mathbf{E}}_n \triangle \mathbf{G}_n^i) \setminus B_{R_n} \right\|_1 = 0 \tag{67}$$

$$\sum_{i=1}^{N} \left\| \mathbf{m}(\hat{\mathbf{E}}_n \triangle \mathbf{E}_n) \setminus B_{r-1}(\mathbf{x}_n^i) \right\|_1 < 6\varepsilon \tag{68}$$

$$\sum_{i=1}^{N} \left\| \mathbf{m}(\mathbf{E}_n \triangle (\mathbf{x}_n^i + \mathbf{y}^i + \mathbf{H}^i) \cap (B_r(\mathbf{x}_n^i)) \setminus B_{r-1}(\mathbf{x}_n^i) \cap B_{R_n} \right\|_1 < 3\varepsilon \tag{69}$$

$$\left\| \mathbf{m}(((\mathbf{x}_n^i + \mathbf{y}^i + \mathbf{H}^i) \triangle \mathbf{G}_n^i) \cap B_{R_n}) \right\|_1 = 0, \tag{70}$$

hence

$$\sum_{i=1}^{N} \left\| \mathbf{m}((\hat{\mathbf{E}}_n \triangle \mathbf{G}_n^i) \cap (B_r(\mathbf{x}_n^i) \setminus B_{r-1}(\mathbf{x}_n^i))) \right\|_1 < 9\varepsilon. \tag{71}$$

For $\rho \in (r-1, r)$ we define

$$\tilde{\mathbf{E}}_n := (\hat{\mathbf{E}}_n \setminus \bigcup_{i=1}^{N} B_\rho(\mathbf{x}_n^i)) \cup \bigcup_{i=1}^{N} (\mathbf{G}_n^i \cap B_\rho(\mathbf{x}_n^i)). \tag{72}$$

By Lemma 3.3, thanks to (71), we can appropriately choose ρ, so that

$$\sum_{i=1}^{N} \operatorname{Per}(\tilde{\mathbf{E}}, \partial B_\rho(\mathbf{x}_n^i)) < 5\varepsilon. \tag{73}$$

Let $\mathbf{M}_n$ be a partition minimizing the perimeter in the family of partitions $\mathbf{F} = (\emptyset, F_2, F_3)$ which coincide with $\hat{\mathbf{E}}_n$ on $\mathbb{R}^d \setminus (B_{R_n} \cap \bigcup_{i=1}^{N} B_r(\mathbf{x}_n^i))$, so that $\mathbf{M}_n \triangle \hat{\mathbf{E}}_n \subseteq \bar{B}_{R_n}$. It follows that

$$\operatorname{Per}(\mathbf{M}_n, \bar{B}_{R_n}) \geq \operatorname{Per}(\mathbf{S}, \bar{B}_{R_n}). \tag{74}$$

Notice that $\tilde{\mathbf{E}}_n$ coincides with $\hat{\mathbf{E}}_n$ outside the intersections $B_{R_n} \cap B_R(\mathbf{x}_n^i)$, hence, by minimality of $\mathbf{M}_n$, we have

$$\operatorname{Per}(\tilde{\mathbf{E}}_n, \bar{B}_{R_n}) \geq \operatorname{Per}(\mathbf{M}_n, \bar{B}_{R_n}). \tag{75}$$

It follows that

$$
\begin{aligned}
\mathrm{Per}(\mathbf{S}, \bar{B}_{R_n}) &\leq \mathrm{Per}(\tilde{\mathbf{E}}_n, \bar{B}_{R_n}) \qquad \text{[by (74) and (75)]}\\
&= \mathrm{Per}(\tilde{\mathbf{E}}_n, B_{R_n}) + \mathrm{Per}(\tilde{\mathbf{E}}_n, \partial B_{R_n})\\
&= \mathrm{Per}(\hat{\mathbf{E}}_n, B_{R_n} \setminus \cup_{i=1}^N \bar{B}_\rho(\mathbf{x}_n^i)) + \mathrm{Per}(\tilde{\mathbf{E}}_n, \partial B_{R_n})\\
&\quad + \sum_{i=1}^N \Big[\mathrm{Per}(\mathbf{G}_n^i, B_\rho(\mathbf{x}_n^i) \cap B_{R_n}) + \mathrm{Per}(\tilde{\mathbf{E}}_n, (\partial B_\rho(\mathbf{x}_n^i)) \cap B_{R_n})\Big] \qquad \text{[by (72)]}\\
&\leq \mathrm{Per}(\hat{\mathbf{E}}_n, B_{R_n} \setminus \cup_{i=1}^N \bar{B}_\rho(\mathbf{x}_n^i)) + \mathrm{Per}(\hat{\mathbf{E}}_n, \partial B_{R_n})\\
&\quad + \sum_{i=1}^N \Big[\mathrm{Per}(\mathbf{G}_n^i, B_\rho(\mathbf{x}_n^i) \cap B_{R_n}) + \mathrm{Per}(\tilde{\mathbf{E}}_n, \partial(B_\rho(\mathbf{x}_n^i) \cap B_{R_n}))\Big]\\
&\leq \mathrm{Per}(\hat{\mathbf{E}}, B_{R_n} \setminus \cup_{i=1}^N \bar{B}_\rho(\mathbf{x}_n^i)) + \mathrm{Per}(\hat{\mathbf{E}}_n, \partial B_{R_n})\\
&\quad + \sum_{i=1}^N \big[\mathrm{Per}(\mathbf{G}_n^i, B_\rho(\mathbf{x}_n^i) \cap B_{R_n}) + \mathrm{Per}(\mathbf{G}_n^i, B_\rho(\mathbf{x}_n^i) \cap \partial B_{R_n})\big] + 5\varepsilon \qquad \text{[by (73)]}\\
&\leq \mathrm{Per}(\mathbf{E}_n, B_{R_n} \setminus \cup_{i=1}^N \bar{B}_\rho(\mathbf{x}_n^i)) + \mathrm{Per}(\mathbf{E}_n, \partial B_{R_n})\\
&\quad + \sum_{i=1}^N \mathrm{Per}(\mathbf{G}_n^i, B_\rho(\mathbf{x}_n^i) \cap \bar{B}_{R_n}) + 5\varepsilon \qquad \text{[by (66), (63)]}\\
&\leq \mathrm{Per}(\mathbf{E}_n, \bar{B}_{R_n})\\
&\quad + \sum_{i=1}^N \big[\mathrm{Per}(\mathbf{G}_n^i, B_\rho(\mathbf{x}_n^i) \cap \bar{B}_{R_n}) - \mathrm{Per}(\mathbf{E}_n, B_\rho(\mathbf{x}_n^i) \cap \bar{B}_{R_n})\big] + 6\varepsilon \qquad \text{[by (65)]}\\
&\leq \mathrm{Per}(\mathbf{E}_n, \bar{B}_{R_n}) \qquad \text{[by (62)]}\\
&\quad + \sum_{i=1}^N \big[\mathrm{Per}(\mathbf{G}_n^i, B_\rho(\mathbf{x}_n^i) \cap \bar{B}_{R_n}) - \mathrm{Per}(\mathbf{F}^i, B_\rho \cap \bar{\Pi}^i)\big] + 6\varepsilon.
\end{aligned}
$$

Hence we get

$$
\begin{aligned}
\Delta_{\mathbf{E}_n} &= \mathrm{Per}(\mathbf{E}_n, \bar{B}_{R_n}) - \mathrm{Per}(\mathbf{S}, \bar{B}_{R_n})\\
&\geq \sum_{i=1}^N \big[\mathrm{Per}(\mathbf{F}^i, B_\rho \cap \bar{\Pi}^i) - \mathrm{Per}(\mathbf{G}_n^i, B_\rho(\mathbf{x}_n^i) \cap \bar{B}_{R_n})\big] - 6\varepsilon\\
&\geq \sum_{i=1}^N \big[\mathrm{Per}(\mathbf{F}^i, B_\rho \cap \bar{\Pi}^i) - \mathrm{Per}(\mathbf{H}^i + \mathbf{y}^i, B_\rho \cap \bar{\Pi}^i)\big] - 7\varepsilon \qquad \text{[by (65)]}\\
&\geq \sum_{i=1}^N \Delta_{\mathbf{F}^i}^{\bar{\Pi}^i} \cdot \big|F_1^i\big|^{\frac{7}{8}} - 8\varepsilon \qquad \text{[by (58)]}\\
&\geq \left(\inf_{i \in \{1,\dots N\}} \Delta_{\mathbf{F}^i}^{\bar{\Pi}^i}\right) \left(\sum_{i=1}^{+\infty} \big|F_1^i\big|^{\frac{7}{8}} - \varepsilon\right) - 8\varepsilon \qquad \text{[by (57)]}\\
&\geq \inf_i \Delta_{\mathbf{F}^i}^{\bar{\Pi}^i}(1 - \varepsilon) - 8\varepsilon,
\end{aligned}
$$

which gives the thesis letting $\varepsilon \to 0$. $\square$

In the following theorem we show that, when Π is a half-space, the blow-down partition in Corollary 4.4 is necessarily flat in Π.

Theorem 5.8 (half-space)**.** *Let $E \subseteq \mathbb{R}^d$ be a locally minimal set in $\mathbb{R}^d$ and suppose there exists a half-space $\Pi \subseteq \mathbb{R}^d$ such that $|E \setminus \Pi| = 0$. Then ∂E is a hyper-plane.*

Proof. Let E_∞ be a blow down of E as given by Corollary 4.4. Clearly $|E_\infty \setminus \Pi| = 0$ and $0 \in \partial E_\infty \cap \partial \Pi$. But Π itself is minimal, hence we can apply the strong maximum principle (see [21, Corollary 1]) to state that $\partial E_\infty = \partial \Pi$. □

Theorem 5.9. *Let $\mathbf{F} = (F_1, F_2, F_3)$ be a 3-partition of $\mathbb{R}^d$ with $|F_1| < +\infty$.*

Assume that Π is a half-space with $0 \in \partial\Pi$ and suppose that each region of $\mathbf{F} \setminus \bar{\Pi}$ is a cone with vertex in 0. In particular $|F_1 \setminus \Pi| = 0$.

Suppose that $\mathbf{F}$ is isoperimetric in $\bar{\Pi}$. Let $\mathbf{F}_\infty$ be a blow-down of $\mathbf{F}$, as in Corollary 4.4.

(i) *If $|F_2 \setminus \Pi| = 0$ or $|F_3 \setminus \Pi| = 0$ then either $\partial \mathbf{F}_\infty = \emptyset$ or $\partial \mathbf{F}_\infty = \partial \Pi$.*
(ii) *If $\mathbf{F} \setminus \bar{\Pi} = (\emptyset, H \setminus \bar{\Pi}, (\mathbb{R}^d \setminus H) \setminus \bar{\Pi})$, where H is a half-space not parallel to $\partial\Pi$ with $0 \in \partial H$ then $\partial \mathbf{F}_\infty \cap \bar{\Pi}$ is a half-hyperplane.*

Proof.

(i) Without loss of generality we can assume that $|F_3 \setminus \Pi| = 0$. Then $\left|F^3_\infty \setminus \Pi\right| = 0$ and $\left|F^1_\infty\right| = 0$. Since $\mathbf{F}_\infty$ is isoperimetric in $\bar{\Pi}$ by Theorem 3.6, $|F_3 \setminus \Pi| = 0$ and Π is convex, then $\partial \mathbf{F}_\infty$ is locally minimal in $\mathbb{R}^d$, and by Theorem 5.8 it is either empty or a hyperplane. In the latter case it necessarily coincides with $\partial\Pi$.
(ii) By Corollary 4.4, $\mathbf{F}_\infty$ is an isoperimetric conical partition in $\bar{\Pi}$. Since $\left|F^1_\infty\right| = 0$ we have $\partial \mathbf{F}_\infty = \partial F^2_\infty$. Consider the $(d-1)$-dimensional rectifiable current T representing the restriction of $\partial \mathbf{F}_\infty$ in $\bar{\Pi}$ (i.e. the current representing the integration on the rectifiable set $\partial F^2_\infty \cap \bar{\Pi}$ with orientation given by the normal vector $\nu_{F^2_\infty}$). Since $\mathbf{F}_\infty$ is locally isoperimetric in $\bar{\Pi}$, and $\bar{\Pi}$ is convex, we know that T is locally minimizing with respect to variations, preserving its boundary ∂T, locally on the whole space $\mathbb{R}^d$. So, ∂T has $C^{1,\alpha}$ regularity and we can apply the boundary regularity result in [10] (see also [3]) which states that T is itself $C^{1,\alpha}$ regular in some neighbourhood of ∂T. In particular if $\partial T \neq 0$ we know that T has a tangent hyperplane in 0 and, being a cone, it must be supported on that hyperplane.

□

Proposition 5.10. *Let Π be a half-space of $\mathbb{R}^d$ with $0 \in \partial\Pi$. Suppose that $\mathbf{F} = (F_1, F_2, F_3)$ is a 3-partition of $\mathbb{R}^d$, isoperimetric in $\bar{\Pi}$, with $|F_1| = |F_1 \cap \Pi| < +\infty$ and $|F_2| = |F_3| = +\infty$, and one of the following:*

(i) *$|F_2 \setminus \Pi| = 0$ or $|F_3 \setminus \Pi| = 0$;*
(ii) *or $\mathbf{F} \setminus \bar{\Pi} = (\emptyset, H \setminus \bar{\Pi}, (\mathbb{R}^d \setminus H) \setminus \bar{\Pi})$, where ∂H is a hyperplane with $0 \in \partial H$.*

Let $\mathbf{F}_\infty$ be any blow-down as in Corollary 4.4. Then $\mathbf{F}_\infty$ is unique, and we have

$$\|\mathbf{m}((\mathbf{F} \triangle (\mathbf{F}_\infty + \mathbf{y})) \cap (B_r \setminus B_{r-1}) \cap \Pi)\| \to 0 \text{ as } r \to +\infty, \text{ for some } \mathbf{y} \in \mathbb{R}^d, \tag{76}$$

$$\Delta^{\bar{\Pi}}_{\mathbf{F}} \geq \Delta_{\mathbf{L}}, \tag{77}$$

where $\mathbf{L}$ is a lens partition.

Proof. The proof is similar in the two cases, we only present the case (ii) which is slightly more difficult. Choose a suitable coordinate system in $\mathbb{R}^d$ such that $\Pi = \{x_d < 0\}$ and $\partial\Pi \cap \partial H = \{x_d = x_{d-1} = 0\}$.
Given a set $G \subseteq \Pi$ we let $\sigma(G) := \{x \in \mathbb{R}^d \setminus \bar{\Pi} : (x_1, \dots, x_{d-2}, -x_{d-1}, -x_d) \notin G\} \subseteq \mathbb{R}^d \setminus \bar{\Pi}$. Using standard density estimates, as in [17, Theorem 2.4], one can prove that there exists $R > 0$ such $|F_1 \setminus B_R| = 0$. Consider now the set

$$E := (F_2 \cap \Pi) \cup \sigma(F_2 \cap \Pi),$$

and observe that E is locally minimal both in $\Pi \setminus \bar{B}_R$ (since $\mathbf{F}$ is isoperimetric and F_1 is contained in B_R) and in $\mathbb{R}^d \setminus (\bar{\Pi} \cup \bar{B}_R)$. Moreover, on $\partial\Pi \setminus \bar{B}_R$ the set ∂E is regular because $\partial\mathbf{F}$ is regular in a neighborhood of $\partial\Pi$. As a consequence $\mathbf{v}(\partial E \setminus \bar{B}_R, 1)$ is a stationary varifold in $\mathbb{R}^d \setminus \bar{B}_R$. By Theorem 4.1 ∂E, hence also $\partial\mathbf{F}$, is asymptotically flat with the expansion in (49), which shows in particular that $\partial\mathbf{F}_\infty \cap \Pi$ is a half-hyperplane and (76) holds. Consider the half-space S such that $S \cap \Pi = \mathbf{F}^2_\infty \cap \Pi$ and consider the Steiner symmetrical set F_1' of F_1 with respect to the hyperplane ∂S. We consider the partition $\mathbf{F}' := (F_1', S \setminus F_1', \mathbb{R}^d \setminus (S \cup F_1'))$ and proceed as in [7, Theorem 2.9]: following [7, Lemma 5.1] one gets $\Delta_{\mathbf{F}'} \le \Delta^{\bar{\Pi}}_{\mathbf{F}}$, then considering the capillarity problem in S with prescribed volume $\frac{1}{2}|F_1|$, one obtains $\Delta_{\mathbf{L}} \le \Delta_{\mathbf{F}'}$. This completes the proof. □

Remark 5.11. We expect that a partition $\mathbf{F}$ as in the proposition above is necessarily a lens partition, intersected with Π. This would follow as in [7, Theorem 2.9] if we knew that $\partial\mathbf{F}_\infty$ is not contained in $\partial\Pi$. On the other hand, there might exist a isoperimetric partition $\mathbf{F}$ in Π with $\partial\mathbf{F}_\infty = \partial\Pi$, which is not a standard lens partition.

We are now ready to prove our main result.

Theorem 5.12 (main result). *There exists $\mathbf{E} = (E_1, E_2, E_3)$ an isoperimetric 3-partition of $\mathbb{R}^8$ with $\mathbf{m}(\mathbf{E}) = (1, +\infty, +\infty)$ which is not a lens partition (Definition 5.3). In particular each blow-down partition $\mathbf{E}_\infty = (\emptyset, E^2_\infty, E^3_\infty)$ is a singular minimal cone.*

Proof. Let $R_n \to +\infty$ be any sequence of positive numbers with $|B_{R_n}| > 1$. Let $\mathbf{E}_n = ((E_n)_1, (E_n)_2, (E_n)_3)$, be an isoperimetric partition in the ball $\bar{B}_{R_n} \subseteq \mathbb{R}^d$ with $|(E_n)_1| = 1$, such that $\mathbf{E} \triangle \mathbf{S} \subseteq \bar{B}_{R_n}$ where $\mathbf{S} = (\emptyset, S_2, S_3)$ is the Simons' partition (see Definition 5.4).

By Theorem 3.1, up to a non-relabeled subsequence, there exist $\mathbf{x}^n_i \in \mathbb{R}^8$ and $\mathbf{F}^i = (F^i_1, F^i_2, F^i_3)$ such that

$$\lim_{n\to+\infty} \left|\mathbf{x}^i_n - \mathbf{x}^j_n\right| = +\infty \qquad \text{if } i \neq j, \tag{78}$$

$$\mathbf{E}_n - \mathbf{x}^i_n \xrightarrow{L^1_{\mathrm{loc}}} \mathbf{F}^i, \qquad \text{as } n \to +\infty, \text{ for all } i \in \mathbb{N} \tag{79}$$

$$\sum_{i\in\mathbb{N}} \left|F^i_1\right| = |(E_n)_1| = 1. \tag{80}$$

Without loss of generality we can assume $\left|F^i_1\right| > 0$ for all i.

Up to a subsequence, we assume that $B_{R_n}(-\mathbf{x}^i_n) \to \Pi^i$ in L^1_{loc} as $n \to +\infty$, where either $\Pi^i = \mathbb{R}^d$ or $\bar{\Pi}^i$ is a half-space of $\mathbb{R}^d$. The limit Π^i cannot be empty because $\bar{\Pi}^i \supset F^i_1$ and we are assuming that $\left|F^i_1\right| > 0$. By Theorem 3.6 for each $i \in \mathbb{N}$ the partition $\mathbf{F}^i$ is isoperimetric in $\bar{\Pi}^i \subseteq \mathbb{R}^d$.

If for some $i \in \mathbb{N}$ we have $\Pi^i = \mathbb{R}^d$, $|F_2^i| = |F_3^i| = +\infty$, and the partition $\mathbf{F}^i$ is not isometric to a lens partition, then the proof is concluded by taking as $\mathbf{E}$ an appropriate rescaling of $\mathbf{F}^i$. Indeed, by [7], an isoperimetric 3-partition of $\mathbb{R}^d$ with only one finite region is a lens partition if its blow-down is a plane, we know that our non-lens partition $\mathbf{E}$ has a blow-down $\mathbf{E}_\infty$ which is a singular minimal cone.

Hence, we assume by contradiction that for all $i \in \mathbb{N}$ either $\mathbf{F}^i$ is a lens partition, or $\Pi^i \neq \mathbb{R}^d$, or $|F_j^i| < +\infty$ for $j = 2$ or $j = 3$. We claim that in any case we have

$$\Delta_{\mathbf{F}^i}^{\bar{\Pi}^i} \geq \Delta_{\mathbf{L}}. \tag{81}$$

If $\mathbf{F}^i$ is a lens partition then (81) trivially holds.

If $\Pi^i \neq \mathbb{R}^d$ it means that the sequence $\mathbf{x}_n^i$ stays close to ∂B_{R_n} and in the limit (as $R_n \to +\infty$) we see an affine half-space. If the sequence also stays close to the cone $\partial \mathbf{S}$, passing to the limit the cone becomes a hyperplane T^i perpendicular to $\partial \Pi^i$ and $\partial \mathbf{F}^i$ coincides with such hyperplane outside $\bar{\Pi}^i$. Without loss of generality we can assume that $0 \in T^i \cap \partial \Pi^i$. We can apply Theorem 5.9, and conclude that the blow-down $\mathbf{F}_\infty^i$ of $\mathbf{F}^i$ is a half-space also inside Π^i, so $\partial \mathbf{F}_\infty^i$ is the union of two half-hyperplanes containing $T^i \cap \partial \Pi^i$. Then we can apply Proposition 5.10 (ii) to obtain (81). Another case is when $\bar{\Pi}^i$ is a half-space, $|F_2^i| = |F_3^i| = +\infty$, and $|F_2^i \setminus \Pi| = 0$ or $|F_3^i \setminus \Pi| = 0$; then by Proposition 5.10 (i) we obtain again (81).

The last case to consider is when either $|F_2^i| < +\infty$ or $|F_3^i| < +\infty$. If for instance $|F_3^i| < +\infty$, it follows that $|F_3^i \setminus \bar{\Pi}^i| = 0$, and $(F_1^i, F_2^i \cup F_3^i)$ is an isoperimetric cluster. In this case F_1^i is a ball B_ρ and we have

$$\Delta_{\mathbf{F}^i} = \Delta_{\mathbf{F}^i}^{\bar{\Pi}^i} = \frac{\mathrm{Per}(B_\rho)}{|B_\rho|^{\frac{7}{8}}} = \frac{8\sqrt{\pi}}{\sqrt[8]{24}} \approx 9.53 > \Delta_{\mathbf{L}}.$$

Hence (81) is proven.

Now we can apply Lemma 5.7 to obtain

$$\limsup_n \Delta_{\mathbf{E}_n} \geq \Delta_{\mathbf{L}}.$$

But when R_n is sufficiently large the barrel partition $\mathbf{Q}$ is a competitor to $\mathbf{E}_n$ and hence we must have $\Delta_{\mathbf{E}_n} \leq \Delta_{\mathbf{Q}}$ which would give $\Delta_{\mathbf{L}} \leq \Delta_{\mathbf{Q}}$ in contrast to the statement of Lemma 5.6. $\square$

Remark 5.13. As observed in the Introduction, an interesting open question is whether $\mathbf{E}_\infty$ coincides with the Simons' cone. We expect that this is the case, but a proof of this fact would require a more refined analysis.

Appendix A. Concentration compactness

In this section we collect some results about concentration compactness for sequences of sets with finite perimeter in the Euclidean space. A generalization of these results were proved in [16] in a more abstract framework.

Lemma A.1 (compactness in L^1_{loc})**.** *Let $E_n \subseteq \mathbb{R}^d$ be a sequence of Caccioppoli sets such that*

$$\sup_{\mathbf{x} \in \mathbb{R}^d} \sup_{n \in \mathbb{N}} \mathrm{Per}(E_n, B_1(\mathbf{x})) < +\infty. \tag{82}$$

Then there exists a Caccioppoli set $E \subseteq \mathbb{R}^d$ such that, up to a subsequence $E_n \to E$ in L^1_{loc}.

Proof. Assumption (82) implies that for all $r > 0$

$$\sup_n \operatorname{Per}(E_n, B_r) < +\infty.$$

So, there is a subsequence such that $E_n \cap B_1$ converges in L^1, up to a subsequence we can assume that also $E_n \cap B_2$ converges, and so on. With a diagonal argument we can assume that $E_n \cap B_k$ converges in L^1 as $n \to +\infty$ for all k, which means that the sequence E_n converge in L^1_{loc}. □

Lemma A.2 (equisummability)**.** *Suppose that* $m_{k,j} \geq 0$*, and*

$$\lim_{n\to+\infty} \sup_{k\in\mathbb{N}} \sum_{j=n}^{+\infty} m_{k,j} = 0.$$

Then

$$\sum_{j=1}^{+\infty} \lim_{k\to+\infty} m_{k,j} = \lim_{k\to+\infty} \sum_{j=0}^{+\infty} m_{k,j}.$$

Theorem A.3 (concentration compactness)**.** *Let* E_k *be a sequence of measurable subsets of* $\mathbb{R}^d$ *such that*

$$\sup_k \operatorname{Per}(E_k) =: P < +\infty, \qquad \limsup_k |E_k| =: m < +\infty. \tag{83}$$

Then, there exist measurable sets F^i*,* $i \in \mathbb{N}$ *and points* $\mathbf{x}^i_k \in \mathbb{R}^d$ *such that, up to a subsequence in* k*, one has*

$$\lim_k \left|\mathbf{x}^i_k - \mathbf{x}^j_k\right| = +\infty \qquad \textit{for all } i \neq j, \tag{84}$$

$$E_k - \mathbf{x}^i_k \xrightarrow{L^1_{\mathrm{loc}}} F^i \qquad \textit{for all } i \textit{ as } k \to +\infty, \tag{85}$$

$$\sum_{i\in\mathbb{N}} |F_i| = \lim_k |E_k|. \tag{86}$$

Proof. Since the statement is invariant under rescaling, without loss of generality we can assume that $m < \frac{1}{2}$ and hence that $|E_k| < \frac{1}{2}$ for every k. For each k let $\mathbf{y}^j_k$ be an enumeration of the lattice $\mathbb{Z}^d$ such that defining $Q^j_k = [0,1]^d + \mathbf{y}^j_k$ the sequence $j \mapsto \left|E_k \cap Q^j_k\right|$ is decreasing. If there are infinitely many cubes in which E_k has positive measure we avoid to enumerate the cubes with 0 measure.

Since $\sup_k P(E_k) < +\infty$, by Lemma A.1, passing to a subsequence, for each j we can suppose that $E_k - \mathbf{y}^j_k$ converges in L^1_{loc} and hence there exists $G^j \subseteq [0,1]^d$ such that

$$(E_k \cap Q^j_k) - y^j_k = (E_k - y^j_k) \cap [0,1]^d \xrightarrow{L^1} G^j, \qquad \text{as } k \to \infty.$$

Note that $\left|G^j\right| = \lim_k \left|E_k \cap Q^j_k\right|$.

Let γ be the local isoperimetric constant, such that if $\left|E \cap [0,1]^d\right| \leq \frac{1}{2}$ then $\left|E \cap [0,1]^d\right| \leq \gamma \operatorname{Per}(E, [0,1]^d)^{1-\frac{1}{d}}$. We have

$$\begin{aligned}\sum_{j=n}^{+\infty} \left|E_k \cap Q_k^j\right| &= \sum_{j=n}^{+\infty} \left|E_k \cap Q_k^j\right|^{\frac{1}{d}} \cdot \left|E_k \cap Q_k^j\right|^{\frac{d-1}{d}} \leq \gamma \sum_{j=n}^{+\infty} \left|E_k \cap Q_k^j\right|^{\frac{1}{d}} \operatorname{Per}(E_k, Q_k^j) \\ &\leq \gamma \left|E_k \cap Q_k^n\right|^{\frac{1}{d}} \sum_{j=n}^{+\infty} \operatorname{Per}(E_k, Q_k^j) \leq \gamma \left(\frac{|E_k|}{n}\right)^{\frac{1}{d}} \sum_{j=n}^{+\infty} \operatorname{Per}(E_k, Q_k^j) \\ &\leq \gamma \left(\frac{|E_k|}{n}\right)^{\frac{1}{d}} \operatorname{Per}(E_k) \leq \frac{\gamma P}{2^{\frac{1}{d}} n^{\frac{1}{d}}}.\end{aligned}$$

Hence

$$\lim_{n \to +\infty} \sup_k \sum_{j=n}^{+\infty} \left|E_k \cap Q_k^j\right| = 0.$$

By Lemma A.2 we have,

$$\sum_j \left|G^j\right| = \sum_j \lim_k \left|E_k \cap Q_k^j\right| = \lim_k \sum_j \left|E_k \cap Q_k^j\right| = \lim_k |E_k|.$$

Up to a subsequence we might suppose that, for all $j, j' \in \mathbb{N}$, the limit $\lim_k \left|\mathbf{y}_k^j - \mathbf{y}_k^{j'}\right|$ exists in $[0, +\infty]$. So we can define an equivalence relation on $\mathbb{N}$, by letting $j \sim j'$ whenever $\lim_k \left|\mathbf{y}_k^j - \mathbf{y}_k^{j'}\right| < +\infty$ and let $I = \mathbb{N}/\sim$ be the quotient set. For each $i \in I$ choose a representant $j_i \in i$.

For every equivalence class i, and $j \in i$, the sequence $\mathbf{y}_k^j - \mathbf{y}_k^{j_i}$ is bounded in $\mathbb{Z}^d$, hence it is constant for k large enough. Up to subsequences we may assume that for every equivalence class i, and $j \in i$, these sequences are constant: $\mathbf{y}_k^j - \mathbf{y}_k^{j_i} = \mathbf{y}^{j,i} \in \mathbb{Z}^d$. Let us denote $\mathbf{y}^{j,i}$ by $\mathbf{y}^j$ if $j \in i$. Clearly for $j \in i$ all $\mathbf{y}^j$ are distinct because $\mathbf{y}_k^j \in \mathbb{Z}^d$ are all distinct. Let $\mathbf{x}_k^i = \mathbf{y}_k^{j_i}$, so that $\mathbf{y}^j = \mathbf{y}_k^j - \mathbf{x}_k^i$.

Let F^i be the L^1_{loc} limit of $E_k - \mathbf{x}_k^i$. Since $(E_k - \mathbf{y}_k^j) \cap [0,1]^d \to G^j$, if $j \in i$

$$\begin{aligned}\left|G^j\right| &= \lim_k \left|E_k \cap Q_k^j\right| = \lim_k \left|E_k \cap ([0,1]^d + \mathbf{y}_k^j)\right| \\ &= \lim_k \left|(E_k - \mathbf{x}_k^i) \cap ([0,1]^d + (\mathbf{y}_k^j - \mathbf{x}_k^i))\right| \\ &= \lim_k \left|(E_k - \mathbf{x}_k^i) \cap ([0,1]^d + \mathbf{y}^j)\right| = \left|F^i \cap ([0,1]^d + \mathbf{y}^j)\right|.\end{aligned}$$

So,

$$\begin{aligned}\lim_k |E_k| &= \sum_j \left|G^j\right| = \sum_i \sum_{j \in i} \left|F^i \cap ([0,1]^d + \mathbf{y}^j)\right| = \sum_i \left|F^i \cap \bigsqcup_{j \in i} ([0,1]^d + \mathbf{y}^j)\right| \\ &\leq \sum_i \left|F^i\right|.\end{aligned}$$

The other inequality follows from the semicontinuity of the measure with respect to the L^1_{loc} convergence. $\square$

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