A fourth-order regularization of the curvature flow of immersed plane curves with Dirichlet boundary conditions

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Abstract

We consider a fourth-order regularization of the curvature flow for an immersed plane curve with fixed boundary, using an elastica-type functional depending on a small positive parameter ε. We show that the approximating flow smoothly converges, as $\varepsilon \to 0^+$, to the curvature flow of the curve with Dirichlet boundary conditions for all times before the first singularity of the limit flow.

Key words: Curvature flow of immersed curves, higher order regularization, Dirichlet boundary conditions

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1 Introduction

In recent years, considerable attention has been devoted to the study of geometric evolution laws for curves and surfaces, especially those governed by curvature-dependent dynamics. It is well established that singularities may develop in finite time during the evolution, a phenomenon which strongly motivates the study of the flow beyond singularities. In the case of mean curvature motion, various definitions of weak solutions have been introduced, beginning with the foundational work of Brakke [2].

Following a suggestion of De Giorgi in [3], we introduce and study a fourth-order regularization of mean curvature flow using an elastica-type functional depending on a small positive parameter ε, which could lead to a new definition of generalized solution. More precisely, given $\varepsilon \in (0,1]$, we consider a time-dependent family of immersed plane curves $\gamma : \bigcup_{t\in[0,T]}(\{t\} \times [0, \ell(\gamma(t))]) \to \mathbb{R}^2$ of class $H^2 \cap C^2$, with fixed boundary points $\gamma(t,0) = P$, $\gamma(t,\ell(\gamma)) = Q$. These curves evolve by the gradient flow of the energy

$$F_\varepsilon(\gamma) := \int_0^{\ell(\gamma)} (1 + \varepsilon\kappa_\gamma^2)ds, \tag{1.1}$$

where, as usual, $ds = |\partial_x\gamma|\ dx$ is the arclength element, κ_γ is the curvature of the curve and $\ell(\gamma)$ its length. In what follows, $\partial_s = |\partial_x\gamma|^{-1}\partial_x$ denotes the differentiation with respect to the

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arclength parameter, while ν_γ and τ_γ denote the oriented normal and tangential vectors of the curve, respectively. Formally $\gamma_\varepsilon(t,x)$ satisfies

$$(\partial_t \gamma_\varepsilon)^\perp = \kappa_{\gamma_\varepsilon(t)}\nu_{\gamma_\varepsilon(t)} - \varepsilon\Big(2\partial_s^2\kappa_{\gamma_\varepsilon(t)} + \kappa^3_{\gamma_\varepsilon(t)}\Big)\nu_{\gamma_\varepsilon(t)} \tag{1.2}$$

coupled with the following boundary and initial conditions:

$$\begin{cases} \gamma_\varepsilon(t,0) = P \\ \gamma_\varepsilon(t,\ell(\gamma_\varepsilon(t))) = Q \\ \kappa_{\gamma_\varepsilon(t)}(0) = \kappa_{\gamma_\varepsilon(t)}(\ell(\gamma_\varepsilon(t))) = 0 \\ \gamma_\varepsilon(0,\cdot) = \overline{\gamma}(\cdot), \end{cases} \tag{1.3}$$

where $\overline{\gamma}$ is an initial regular immersion such that, for some $\alpha \in (0,1)$,

$$\overline{\gamma} \in C^{4+\alpha}([0,\ell(\overline{\gamma})];\mathbb{R}^2) \quad \text{with} \quad \overline{\gamma}(0) = P, \quad \overline{\gamma}(\ell(\overline{\gamma})) = Q, \tag{1.4}$$

and

$$\kappa_{\overline{\gamma}}(0) = 0 = \kappa_{\overline{\gamma}}(\ell(\overline{\gamma})). \tag{1.5}$$

We point out that, also if we assume that $\overline{\gamma}$ is an injective curve (with no self-intersections), since the flow does not preserve embeddedness, each γ_ε is just an immersion (i.e. possibly self-intersecting image). Furthermore, even if $\overline{\gamma}$ coincides with the graph of some function $f : [a,b] \to \mathbb{R}$, curves γ_ε cannot be assumed to be graphs as well, since the gradient flow of F_ε is a fourth order parabolic system, which in general does not preserve graphicality.
We can regard (1.2) as a perturbation of the curvature flow, coinciding with it when $\varepsilon = 0$.
In [7, Theorem 2.1] [5, Theorem 4.15] it is proven that, given $\varepsilon \in (0,1]$ and $\overline{\gamma}$ a planar immersed regular curve satisfying (1.4), (1.5), then the initial boundary value problem formed by (1.2), (1.3) admits a unique smooth solution γ_ε defined for all times, that is

$$\gamma_\varepsilon \in C^{\frac{4+\alpha}{4},4+\alpha}\Big(\bigcup_{t\in[0,+\infty)}(\{t\}\times[0,\ell(\gamma_\varepsilon(t))])\,;\mathbb{R}^2\Big) \cap C^\infty\Big(\bigcup_{t\in[\delta,+\infty)}(\{t\}\times[0,\ell(\gamma_\varepsilon(t))])\,;\mathbb{R}^2\Big) \qquad \forall\delta>0.$$

Therefore our focus is on the convergence to the curvature flow as $\varepsilon \to 0^+$.
Let us state our main result. Denote by γ the immersed curvature flow of $\overline{\gamma}$ solution to

$$\begin{cases} (\partial_t\gamma)^\perp = \kappa_{\gamma(t)}\nu_{\gamma(t)} \\ \gamma(t,0) = P, \quad \gamma(t,\ell(\gamma(t))) = Q \\ \gamma(0,\cdot) = \overline{\gamma}(\cdot). \end{cases} \tag{1.6}$$

It is possible to show that (1.6) admits a unique solution in a maximal time interval $[0,T_{\text{sing}})$, with $T_{\text{sing}} \in (0,+\infty]$. Indeed, given $\alpha \in (0,1)$ and $\overline{\gamma} \in C^{2+\alpha}([0,\ell(\overline{\gamma})];\mathbb{R}^2)$ a planar immersed curve satisfying $\overline{\gamma}(0) = P$, $\overline{\gamma}(\ell(\overline{\gamma})) = Q$, then there exists $T_{\text{sing}} > 0$ such that the initial boundary value problem (1.6) admits a unique solution belonging to

$$C^{\frac{4+2\alpha}{4},2+\alpha}\Big(\bigcup_{t\in[0,T_{\text{sing}})}(\{t\}\times[0,\ell(\gamma(t))])\,;\mathbb{R}^2\Big) \cap C^\infty\Big(\bigcup_{t\in[\delta,T_{\text{sing}})}(\{t\}\times[0,\ell(\gamma(t))])\,;\mathbb{R}^2\Big) \qquad \forall\delta\in(0,T_{\text{sing}})$$

(see also [4] for a related statement). In addition, one checks that the condition (1.5) is preserved during the flow.
Given an immersed curve γ (resp. $\gamma(t)$, γ_ε, $\gamma_\varepsilon(t)$) parametrized by arc-length, in what follows we indicate by Υ (resp. $\Upsilon(t)$, Υ_ε, $\Upsilon_\varepsilon(t)$) the constant speed reparametrization of γ (resp. $\gamma(t)$, γ_ε, $\gamma_\varepsilon(t)$) on the interval $[0,1]$.
Our main convergence result reads as follows.

Theorem 1.1 (**Asymptotic convergence**). *Let $\overline{\gamma}$ be a planar immersed regular curve satisfying* (1.4), (1.5)*, and let γ be the solution to* (1.6) *in* $[0, T_{\mathrm{sing}})$*. For any $\varepsilon \in (0,1]$ denote by γ_ε the solution to* (1.2), (1.3) *in* $[0,+\infty)$*. Then the corresponding parametrizations Υ_ε converge in $C^\infty_{\mathrm{loc}}((0,T_{\mathrm{sing}}) \times [0,1])$ to the map Υ, as $\varepsilon \to 0^+$. Furthermore, if $\overline{\gamma} \in C^\infty([0,\ell(\overline{\gamma})])$ and all derivatives of $\overline{\gamma}$ of even order at* 0 *and $\ell(\overline{\gamma})$ vanish, then the convergence takes place in $C^\infty_{\mathrm{loc}}([0,T_{\mathrm{sing}}) \times [0,1])$.*

As a consequence, we observe that for $t \in (0, T_{\mathrm{sing}})$, all space derivatives of $\gamma(t,\cdot)$ of even order at 0 and $\ell(\gamma(t))$ vanish.
The proof of Theorem 1.1 partially follows the lines in [1], where convergence is established for the curvature flow of regular *closed* curves immersed in $\mathbb{R}^2$. The novelty in the present setting lies in accounting for boundary terms and for the tangential component of the velocity, that we denote with $\lambda_\varepsilon := \langle \partial_t \gamma_\varepsilon, \tau_{\gamma_\varepsilon}\rangle$.
The crucial point is to obtain ε-independent integral estimates of the curvature and all its derivatives. Indeed, one can see that, by an ODE's argument, since all the flows (letting $0 < \varepsilon \le 1$ vary) start from a common initial smooth curve, fixing any $j \in \mathbb{N}$, there exists a common positive interval of time such that all the quantities $\|\partial_s^i \kappa\|_{L^2}$, for $i \in \{0,\dots,j\}$ are equibounded. This will allow us to get compactness and C^∞ convergence to the curvature flow as $\varepsilon \to 0^+$. Furthermore, our convergence result holds up to T_{sing}, and not only up to $T_{\max}$ (defined in Proposition 3.5).
We point out that the case of a fourth-order regularization of the curvature flow for immersed planar curves with different boundary conditions is currently under investigation. For examples, Neumann boundary conditions, or when the boundary points are free to move along two parallel lines, and even, more generally, with prescribed trajectories. However, this analysis appears to be significantly more challenging than the Dirichlet case, because in the latter case, one can exploit the fact that the velocity $\partial_t \gamma$ vanishes at the boundary. This allows one to discard all boundary terms arising from integration by parts. In contrast, under different boundary conditions, we currently do not know how to handle these contributions, which are related with the presence of tangential velocity at the boundary points, a quantity that in general seems difficult to estimate.

2 Notation and preliminaries

Given $I := [0,1]$, we consider planar parametrized immersed curves $\Upsilon : I \to \mathbb{R}^2$. If $\Upsilon \in C^k(I,\mathbb{R}^2)$ we say that Υ is of class C^k. A curve of class C^1 is regular if $\Upsilon_x(x) = \frac{d\Upsilon}{dx}(x) \neq 0$ for every $x \in I$, in which case its unit tangent vector $\tau = \tau_\gamma = \Upsilon_x/|\Upsilon_x|$ is well-defined. We indicate by $\nu = \nu_\gamma$ the unit normal vector, which is the counterclockwise rotation R by $\pi/2$ of τ, $\nu = R\tau$. The arc-length parameter of curve Υ (null in zero) is denoted by s and is given by

$$s := s(x) = \int_0^x |\Upsilon_x(\zeta)| d\zeta, \qquad x \in I.$$

The curve Υ, reparametrized by arc-length, will be indicated by $\gamma : [0,\ell(\gamma)] \to \mathbb{R}^2$, with $\tau_\gamma = \gamma_s$ and

$$\ell(\gamma) := \int_I |\Upsilon_x(x)| dx$$

the length of γ. If γ is of class C^2, we define the curvature vector $\gamma_{ss} =: \kappa_\gamma \nu_\gamma$. We also recall that

$$\partial_s \nu_\gamma = -\kappa_\gamma \tau_\gamma. \tag{2.1}$$

In what follows we will consider time-dependent families of curves $(\Upsilon(t,x))_{t\in[0,T]}$. We often write $\gamma(t,\cdot) = \gamma(t)(\cdot)$. Again, we let $\tau_{\gamma(t)}$ be the unit tangent vector to the curve, $\nu_{\gamma(t)}$ the unit normal vector and $\kappa_{\gamma(t)}$ its curvature.

We denote by $\partial_x f$, $\partial_s f$ and $\partial_t f$ the derivatives of a function f along a curve with respect to the variable x, the arc length parameter s on such a curve and the time, respectively. Moreover $\partial_x^n f$, $\partial_s^n f$, $\partial_t^n f$ are the higher order partial derivatives.
When we parametrize γ by constant speed on the interval $[0,1]$ we have $s(t,x) = x\ell(\gamma(t))$, and $|\partial_x \Upsilon(t)| = \ell(\gamma(t))$.
In what follows, with a small abuse of notation, we sometimes write $\int_\gamma (1+\varepsilon\kappa^2)ds$ in place of the right-hand side of (1.1). A similar notation will be adopted for integrals of general quantities on a curve γ.

3 Estimates of geometric quantities

Let us denote

$$E_\varepsilon(t,\cdot) = E_\varepsilon := -\kappa_{\gamma_\varepsilon(t)} + \varepsilon\Big(2\partial_s^2\kappa_{\gamma_\varepsilon(t)} + \kappa^3_{\gamma_\varepsilon(t)}\Big) \tag{3.1}$$

the normal velocity of a curve γ_ε evolving by the ε-elastic flow. In (1.2) only the normal component of the velocity is prescribed. This does not mean that the tangential velocity is necessarily zero. Indeed the motion equation can be written as

$$\partial_t\gamma_\varepsilon = -E_\varepsilon\nu_{\gamma_\varepsilon(t)} + \lambda_\varepsilon\tau_{\gamma_\varepsilon(t)} \quad \text{in } \mathrm{dom}(\gamma_\varepsilon) := \bigcup_{t\in[0,+\infty)} \left(\{t\}\times[0,\ell(\gamma_\varepsilon(t))]\right), \tag{3.2}$$

where

$$\lambda_\varepsilon = \langle \partial_t\gamma_\varepsilon, \tau_{\gamma_\varepsilon}\rangle.$$

We observe that from the Dirichlet boundary conditions in (1.3), it follows $\frac{d}{dt}\gamma_\varepsilon(t,s) = 0$ for $s \in \{0, \ell(\gamma_\varepsilon(t))\}$. Therefore, from the evolution equation (3.2) we deduce

$$E_\varepsilon(t,0) = 0, \quad \lambda_\varepsilon(t,0) = 0 \quad \forall t \in (0,+\infty) \tag{3.3}$$

and

$$\begin{aligned} 0 &= \partial_t\gamma_\varepsilon(t,\ell(\gamma_\varepsilon(t))) + \partial_s\gamma_\varepsilon(t,\ell(\gamma_\varepsilon(t)))\dot\ell(\gamma_\varepsilon(t)) \\ &= -E_\varepsilon(t,\ell(\gamma_\varepsilon(t)))\nu_{\gamma_\varepsilon(t)}(\ell(\gamma_\varepsilon(t))) + [\lambda_\varepsilon(t,\ell(\gamma_\varepsilon(t))) + \dot\ell(\gamma_\varepsilon(t))]\tau_{\gamma_\varepsilon(t)}(\ell(\gamma_\varepsilon(t))), \end{aligned}$$

from which we get

$$E_\varepsilon(t,\ell(\gamma_\varepsilon(t))) = 0 \qquad \forall t \in (0,+\infty), \tag{3.4}$$

and

$$\lambda_\varepsilon(t,\ell(\gamma_\varepsilon(t))) = -\dot\ell(\gamma_\varepsilon(t)) \qquad \forall t \in (0,+\infty). \tag{3.5}$$

We recall from [5, Lemma 2.19] the following formulas, which we prove for completeness.

Lemma 3.1 (Evolution of geometric quantities)**.** *Let $\varepsilon \in (0,1]$ and $\gamma_\varepsilon : dom(\gamma_\varepsilon) \to \mathbb{R}^2$ be a curve moving by* (3.2)*. Then the following formulas hold:*

$$\partial_t\partial_s - \partial_s\partial_t = (-\kappa_{\gamma_\varepsilon}E_\varepsilon - \partial_s\lambda_\varepsilon)\partial_s, \tag{3.6}$$

$$\partial_t(ds) = (\kappa_{\gamma_\varepsilon}E_\varepsilon + \partial_s\lambda_\varepsilon)\ ds, \tag{3.7}$$

$$\partial_t\tau_{\gamma_\varepsilon} = (-\partial_s E_\varepsilon + \lambda_\varepsilon\kappa_{\gamma_\varepsilon})\nu_{\gamma_\varepsilon}, \tag{3.8}$$

$$\partial_t\nu_{\gamma_\varepsilon} = -(-\partial_s E_\varepsilon + \lambda_\varepsilon\kappa_{\gamma_\varepsilon})\tau_{\gamma_\varepsilon}, \tag{3.9}$$

and

$$\begin{aligned} \partial_t\kappa_{\gamma_\varepsilon} &= -\partial_s^2 E_\varepsilon - \kappa^2_{\gamma_\varepsilon}E_\varepsilon + \lambda_\varepsilon\partial_s\kappa_{\gamma_\varepsilon} \\ &= \partial_s^2\kappa_{\gamma_\varepsilon(t)} + \kappa^3_{\gamma_\varepsilon(t)} - 2\varepsilon\partial_s^4\kappa_{\gamma_\varepsilon(t)} - 6\varepsilon\kappa_{\gamma_\varepsilon(t)}(\partial_s\kappa_{\gamma_\varepsilon(t)})^2 - 5\varepsilon\kappa^2_{\gamma_\varepsilon(t)}\partial_s^2\kappa_{\gamma_\varepsilon(t)} - \varepsilon\kappa^5_{\gamma_\varepsilon(t)} \\ &\quad + \lambda_\varepsilon\partial_s\kappa_{\gamma_\varepsilon}. \end{aligned} \tag{3.10}$$

Proof. We have

$$\begin{aligned}\partial_t\partial_s - \partial_s\partial_t &= \frac{\partial_t\partial_x}{|\partial_x\gamma_\varepsilon|} - \frac{\langle\partial_x\gamma_\varepsilon, \partial_t\partial_x\gamma_\varepsilon\rangle\partial_x}{|\partial_x\gamma_\varepsilon|^3} - \frac{\partial_x\partial_t}{|\partial_x\gamma_\varepsilon|} = -\langle\tau_{\gamma_\varepsilon}, \partial_s\partial_t\gamma_\varepsilon\rangle\partial_s \\ &= -\langle\tau_{\gamma_\varepsilon}, \partial_s(-E_\varepsilon\nu_{\gamma_\varepsilon} + \lambda_\varepsilon\tau_{\gamma_\varepsilon})\rangle\partial_s = (-\kappa_{\gamma_\varepsilon}E_\varepsilon - \partial_s\lambda_\varepsilon)\partial_s,\end{aligned}$$

where in the last equality we used equation (2.1). Thus (3.6) holds, and we can compute

$$\begin{aligned}\partial_t(ds) &= \partial_t|\partial_x\gamma_\varepsilon|dx = \frac{\langle\partial_x\gamma_\varepsilon, \partial_t\partial_x\gamma_\varepsilon\rangle}{|\partial_x\gamma_\varepsilon|}dx = \langle\tau_{\gamma_\varepsilon}, \partial_s\partial_t\gamma_\varepsilon\rangle ds = (\kappa_{\gamma_\varepsilon}E_\varepsilon + \partial_s\lambda_\varepsilon)ds, \\ \partial_t\tau_{\gamma_\varepsilon} &= \partial_t\partial_s\gamma_\varepsilon = \partial_s\partial_t\gamma_\varepsilon - (\kappa_{\gamma_\varepsilon}E_\varepsilon + \partial_s\lambda_\varepsilon)\partial_s\gamma_\varepsilon \\ &= \partial_s(-E_\varepsilon\nu_{\gamma_\varepsilon} + \lambda_\varepsilon\tau_{\gamma_\varepsilon}) - (\kappa_{\gamma_\varepsilon}E_\varepsilon + \partial_s\lambda_\varepsilon)\tau_{\gamma_\varepsilon} = (-\partial_sE_\varepsilon + \kappa_{\gamma_\varepsilon}\lambda_\varepsilon)\nu_{\gamma_\varepsilon}.\end{aligned}$$

Moreover $\partial_t\nu_{\gamma_\varepsilon} = \partial_t(R\tau_{\gamma_\varepsilon}) = R(\partial_t\tau_{\gamma_\varepsilon}) = -(-\partial_sE_\varepsilon + \kappa_{\gamma_\varepsilon}\lambda_\varepsilon)\tau_{\gamma_\varepsilon}$, and

$$\begin{aligned}\partial_t\kappa_{\gamma_\varepsilon} &= \partial_t\langle\partial_s\tau_{\gamma_\varepsilon}, \nu_{\gamma_\varepsilon}\rangle = \langle\partial_t\partial_s\tau_{\gamma_\varepsilon}, \nu_{\gamma_\varepsilon}\rangle = \langle\partial_s\partial_t\tau_{\gamma_\varepsilon}, \nu_{\gamma_\varepsilon}\rangle + (-\kappa_{\gamma_\varepsilon}E_\varepsilon - \partial_s\lambda_\varepsilon)\langle\partial_s\tau_{\gamma_\varepsilon}, \nu_{\gamma_\varepsilon}\rangle \\ &= \partial_s\langle\partial_t\tau_{\gamma_\varepsilon}, \nu_{\gamma_\varepsilon}\rangle - \kappa_{\gamma_\varepsilon}^2E_\varepsilon - \kappa_{\gamma_\varepsilon}\partial_s\lambda_\varepsilon = \partial_s(-\partial_sE_\varepsilon + \lambda_\varepsilon\kappa_{\gamma_\varepsilon}) - \kappa_{\gamma_\varepsilon}^2E_\varepsilon - \kappa_{\gamma_\varepsilon}\partial_s\lambda_\varepsilon \\ &= -\partial_s^2E_\varepsilon + \lambda_\varepsilon\partial_s\kappa_{\gamma_\varepsilon} - \kappa_{\gamma_\varepsilon}^2E_\varepsilon.\end{aligned}$$

By expanding E_ε, the second equality in (3.10) follows. □

Fixing $s \in (0, \ell(\gamma_\varepsilon(t)))$, and using (3.7) we have

$$\begin{aligned}0 = \partial_t\int_0^s d\sigma &= \int_0^s \partial_s\lambda_\varepsilon(t,\sigma)\ d\sigma + \int_0^s E_\varepsilon\kappa_{\gamma_\varepsilon(t)}\ d\sigma \\ &= \lambda_\varepsilon(t,s) - \lambda_\varepsilon(t,0) + \int_0^s E_\varepsilon\kappa_{\gamma_\varepsilon(t)}\ d\sigma\end{aligned}$$

from which, recalling (3.3), we get

$$\lambda_\varepsilon(t,s) = -\int_0^s E_\varepsilon\kappa_{\gamma_\varepsilon(t)}\ d\sigma. \tag{3.11}$$

3.1 ε-uniform estimates of length, energy and normal velocity

Obviously

$$\ell(\gamma_\varepsilon(t)) \geq |P-Q| \qquad \forall t \in [0,+\infty),\ \forall\varepsilon \in (0,1]. \tag{3.12}$$

Moreover there exists a constant $C = C(\overline{\gamma}) > 0$ such that

$$\sup_{t\in[0,+\infty)}\sup_{\varepsilon\in(0,1]}\ell(\gamma_\varepsilon(t)) \leq C \qquad \forall\varepsilon \in (0,1]. \tag{3.13}$$

Indeed,

$$\ell(\gamma_\varepsilon(t)) \leq F_\varepsilon(\gamma_\varepsilon(t)) \leq F_\varepsilon(\overline{\gamma}) \leq \int_0^{\ell(\overline{\gamma})}\left(1 + \kappa_{\overline{\gamma}}^2\right)\ ds \leq C,$$

as a consequence of the fact that the PDE in (1.2) is the gradient flow of F_ε and $\varepsilon \in (0,1]$.

Lemma 3.2 (Energy dissipation equality)**.** *For any $\varepsilon \in (0,1]$ and any $t \in (0,+\infty)$ we have*

$$\begin{aligned}\frac{d}{dt}F_\varepsilon(\gamma_\varepsilon(t)) &= -\int_0^{\ell(\gamma_\varepsilon(t))} E_\varepsilon^2(t,s)ds \\ &= -\frac{1}{2}\int_0^{\ell(\gamma_\varepsilon(t))} |(\partial_t\gamma_\varepsilon(t))^\perp|^2ds - \frac{1}{2}\int_0^{\ell(\gamma_\varepsilon(t))} E_\varepsilon^2(t,s)ds.\end{aligned} \tag{3.14}$$

Proof. Using (3.7), (3.10) and (3.5), we get

$$\begin{aligned}
\frac{d}{dt}F_\varepsilon(\gamma_\varepsilon(t))) =& \frac{d}{dt}\int_0^{\ell(\gamma_\varepsilon(t))}\Big(1+\varepsilon\kappa^2_{\gamma_\varepsilon(t)}\Big)\,ds\\
=& \int_0^{\ell(\gamma_\varepsilon(t))}\Big(2\varepsilon\kappa_{\gamma_\varepsilon(t)}\partial_t\kappa_{\gamma_\varepsilon(t)}+(1+\varepsilon\kappa^2_{\gamma_\varepsilon(t)})(\kappa_{\gamma_\varepsilon(t)}E_\varepsilon+\partial_s\lambda_\varepsilon)\Big)\,ds\\
&+\Big(1+\varepsilon\kappa^2_{\gamma_\varepsilon(t)}(\ell(\gamma_\varepsilon(t)))\Big)\dot\ell(\gamma_\varepsilon(t))\\
=& \int_0^{\ell(\gamma_\varepsilon(t))}\Big(2\varepsilon\kappa_{\gamma_\varepsilon(t)}(-\partial_s^2E_\varepsilon-\kappa^2_{\gamma_\varepsilon}E_\varepsilon+\lambda_\varepsilon\partial_s\kappa_{\gamma_\varepsilon(t)})+(1+\varepsilon\kappa^2_{\gamma_\varepsilon(t)})(\kappa_{\gamma_\varepsilon(t)}E_\varepsilon+\partial_s\lambda_\varepsilon)\Big)ds\\
&-\lambda_\varepsilon(t,\ell(\gamma_\varepsilon(t)))\Big(1+\varepsilon\kappa^2_{\gamma_\varepsilon(t)}(\ell(\gamma_\varepsilon(t)))\Big)\\
=& \int_0^{\ell(\gamma_\varepsilon(t))}\Big(-2\varepsilon\kappa_{\gamma_\varepsilon(t)}\partial_s^2E_\varepsilon-E_\varepsilon(\varepsilon\kappa^3_{\gamma_\varepsilon(t)}-\kappa_{\gamma_\varepsilon(t)})+\partial_s(\lambda_\varepsilon(1+\varepsilon\kappa^2_{\gamma_\varepsilon(t)}))\Big)ds\\
&-\lambda_\varepsilon(t,\ell(\gamma_\varepsilon(t)))\Big(1+\varepsilon\kappa^2_{\gamma_\varepsilon(t)}(\ell(\gamma_\varepsilon(t)))\Big).
\end{aligned}$$

Integrating twice by parts the term $-2\varepsilon\kappa_{\gamma_\varepsilon(t)}\partial_s^2E_\varepsilon$ and observing that

$$-2\varepsilon\partial_s^2\kappa_{\gamma_\varepsilon(t)}E_\varepsilon-E_\varepsilon(\varepsilon\kappa^3_{\gamma_\varepsilon(t)}-\kappa_{\gamma_\varepsilon(t)})=-E_\varepsilon(2\varepsilon\partial_s^2\kappa_{\gamma_\varepsilon(t)}+\varepsilon\kappa^3_{\gamma_\varepsilon(t)}-\kappa_{\gamma_\varepsilon(t)})=-E_\varepsilon^2,$$

we obtain

$$\frac{d}{dt}F_\varepsilon(\gamma_\varepsilon(t))=-\int_0^{\ell(\gamma_\varepsilon(t))}E_\varepsilon^2\,ds+2\varepsilon\Big[-\kappa_{\gamma_\varepsilon(t)}\partial_sE_\varepsilon+\partial_s\kappa_{\gamma_\varepsilon(t)}E_\varepsilon\Big]_0^{\ell(\gamma_\varepsilon(t))}-\lambda_\varepsilon(t,0)\Big(1+\varepsilon\kappa^2_{\gamma_\varepsilon(t)}(0)\Big),$$

where we recall that $\gamma_\varepsilon\in C^\infty(\mathrm{dom}(\gamma_\varepsilon);\mathbb{R}^2)$ for any $\delta>0$.
It remains to show that

$$2\varepsilon\Big[-\kappa_{\gamma_\varepsilon(t)}\partial_sE_\varepsilon(t,\cdot)+\partial_s\kappa_{\gamma_\varepsilon(t)}E_\varepsilon(t,\cdot)\Big]_0^{\ell(\gamma_\varepsilon(t))}-\lambda_\varepsilon(t,0)\Big(1+\varepsilon\kappa^2_{\gamma_\varepsilon(t)}(0)\Big)=0. \tag{3.15}$$

From (1.3) we have

$$\kappa_{\gamma_\varepsilon(t)}(s)=0 \quad \text{for } s\in\{0,\ell(\gamma_\varepsilon(t))\}$$

which, together with (3.3) and (3.4), yields (3.15). □

Remark 3.3. In the proof of Lemma 3.2 we heavily use the Dirichlet boundary conditions. Considering other boundary conditions would present the challenge of handling the boundary terms in equation (3.15).

Corollary 3.4 (L^2-estimate of the normal velocity)**.** *There exists a constant $C=C_{\overline{\gamma}}>0$ such that for any $T\in(0,+\infty)$*

$$\sup_{\varepsilon\in(0,1]}\int_0^T\|(\partial_t\gamma_\varepsilon(t))^\perp\|^2_{L^2([0,\ell(\gamma_\varepsilon(t))];\mathbb{R}^2)}\,dt\le C. \tag{3.16}$$

Proof. From Lemma 3.2, integrating over time, we have

$$\int_0^T\frac{d}{dt}F_\varepsilon(\gamma_\varepsilon(t))\,dt=-\int_0^T\int_0^{\ell(\gamma_\varepsilon(t))}(E_\varepsilon)^2ds$$

from which

$$F_\varepsilon(\gamma_\varepsilon(T,\cdot))+\int_0^T\int_0^{\ell(\gamma_\varepsilon(t))}E_\varepsilon^2ds\,dt=F_\varepsilon(\gamma_\varepsilon(0,\cdot))=F_\varepsilon(\overline{\gamma})\le\int_0^{\ell(\overline{\gamma})}\Big(1+\kappa^2_{\overline{\gamma}}\Big)\,ds\le C,$$

where we have used the initial condition in (1.3). As a consequence

$$\int_0^T\int_0^{\ell(\gamma_\varepsilon(t))}E_\varepsilon^2ds\,dt=\int_0^T\|(\partial_t\gamma_\varepsilon(t))^\perp\|^2_{L^2([0,\ell(\gamma_\varepsilon(t))];\mathbb{R}^2)}\,dt\le C.$$

□

3.2 ε-uniform estimate of $\|\kappa_{\gamma_\varepsilon}\|_2$

Let γ_ε be the solution to (1.2), (1.3). One of the crucial ingredients to prove Theorem 1.1 is the following

Proposition 3.5. *There exists $T_{\max} > 0$ such that, for any $T \in (0, T_{\max})$*

$$\sup_{\varepsilon\in(0,1]} \sup_{t\in[0,T]} \int_0^{\ell(\gamma_\varepsilon(t))} \kappa^2_{\gamma_\varepsilon(t)}\, ds < +\infty. \tag{3.17}$$

In order to prove this proposition 3.5 we need some preliminaries. Although the statement of the next lemma coincides with that of [1, Lemma 3.9], it must be reproven in our setting, as it requires taking into account both the boundary contributions and the tangential component of the velocity.

Lemma 3.6. *For any $t \in (0, +\infty)$ we have*

$$\begin{aligned} \frac{d}{dt}\int_0^{\ell(\gamma_\varepsilon(t))} \kappa^2_{\gamma_\varepsilon(t)} ds = &\int_0^{\ell(\gamma_\varepsilon(t))} \Big(-2(\partial_s \kappa_{\gamma_\varepsilon(t)})^2 + \kappa^4_{\gamma_\varepsilon(t)}\Big) ds \\ &+ \varepsilon \int_0^{\ell(\gamma_\varepsilon(t))} \Big(-4(\partial_s^2(\kappa_{\gamma_\varepsilon(t)}))^2 - \kappa^6_{\gamma_\varepsilon(t)} - 4\kappa^3_{\gamma_\varepsilon(t)}\partial_s^2\kappa_{\gamma_\varepsilon(t)}\Big) ds. \end{aligned} \tag{3.18}$$

Proof. Using equality (3.7), we get

$$\begin{aligned} &\frac{d}{dt}\int_0^{\ell(\gamma_\varepsilon(t))} \kappa^2_{\gamma_\varepsilon(t)}\, ds \\ &= \int_0^{\ell(\gamma_\varepsilon(t))} 2\kappa_{\gamma_\varepsilon(t)}\partial_t\kappa_{\gamma_\varepsilon(t)}\, ds + \int_0^{\ell(\gamma_\varepsilon(t))} \Big(-\kappa^4_{\gamma_\varepsilon(t)} + 2\varepsilon\kappa^3_{\gamma_\varepsilon(t)}\partial_s^2\kappa_{\gamma_\varepsilon(t)} + \varepsilon\kappa^6_{\gamma_\varepsilon(t)} + \kappa^2_{\gamma_\varepsilon(t)}\partial_s\lambda_\varepsilon\Big) ds \\ &\quad + \kappa^2_{\gamma_\varepsilon(t)}(\ell(\gamma_\varepsilon(t)))\dot{\ell}(\gamma_\varepsilon(t)). \end{aligned}$$

From (1.3) we have $\kappa_{\gamma_\varepsilon(t)}(\ell(\gamma_\varepsilon(t))) = 0$, thus the term $\kappa^2_{\gamma_\varepsilon(t)}(\ell(\gamma_\varepsilon(t)))\dot{\ell}(\gamma_\varepsilon(t)) = 0$. Using (3.10), and coupling together the terms $2\kappa_{\gamma_\varepsilon(t)}\lambda_\varepsilon\partial_s\kappa_{\gamma_\varepsilon(t)}$ and $\kappa^2_{\gamma_\varepsilon(t)}\partial_s\lambda_\varepsilon$, we obtain

$$\begin{aligned} &\frac{d}{dt}\int_0^{\ell(\gamma_\varepsilon(t))} \kappa^2_{\gamma_\varepsilon(t)}\, ds \\ &\quad= \int_0^{\ell(\gamma_\varepsilon(t))} \Big(2\kappa_{\gamma_\varepsilon(t)}\partial_s^2\kappa_{\gamma_\varepsilon(t)} + \kappa^4_{\gamma_\varepsilon(t)}\Big) ds \\ &\quad\quad+ \varepsilon\int_0^{\ell(\gamma_\varepsilon(t))} \Big(-4\kappa_{\gamma_\varepsilon(t)}\partial_s^4\kappa_{\gamma_\varepsilon(t)} - 12\kappa^2_{\gamma_\varepsilon(t)}(\partial_s\kappa_{\gamma_\varepsilon(t)})^2 - 8\kappa^3_{\gamma_\varepsilon(t)}\partial_s^2\kappa_{\gamma_\varepsilon(t)} - \kappa^6_{\gamma_\varepsilon(t)}\Big) ds \\ &\quad\quad+ \int_0^{\ell(\gamma_\varepsilon(t))} \partial_s(\lambda_\varepsilon\kappa^2_{\gamma_\varepsilon(t)})\, ds. \end{aligned} \tag{3.19}$$

Again from (1.3) we deduce that $\displaystyle\int_0^{\ell(\gamma_\varepsilon(t))} \partial_s(\lambda_\varepsilon\kappa^2_{\gamma_\varepsilon(t)})\, ds = 0$.

We now argue as in [1] with the difference to be pointed out here that we need to take into account the boundary terms. The first and third terms on the right hand side of (3.19) depend on higher derivatives and, moreover, they lack a sign. Therefore, we integrate by parts (with respect to s) once for the term without ε and twice for the term with ε obtaining:

$$\begin{aligned}
&\frac{d}{dt}\int_0^{\ell(\gamma_\varepsilon(t))} \kappa^2_{\gamma_\varepsilon(t)}\, ds \\
&\quad= \int_0^{\ell(\gamma_\varepsilon(t))} \Big(-2(\partial_s \kappa_{\gamma_\varepsilon(t)})^2 + \kappa^4_{\gamma_\varepsilon(t)}\Big) ds \\
&\qquad + \varepsilon \int_0^{\ell(\gamma_\varepsilon(t))} \Big(-4(\partial_s^2 \kappa_{\gamma_\varepsilon(t)})^2 - 12\kappa^2_{\gamma_\varepsilon(t)}(\partial_s\kappa_{\gamma_\varepsilon(t)})^2 - 8\kappa^3_{\gamma_\varepsilon(t)}\partial_s^2\kappa_{\gamma_\varepsilon(t)} - \kappa^6_{\gamma_\varepsilon(t)}\Big) ds \\
&\qquad + \Big[2\kappa_{\gamma_\varepsilon(t)}\partial_s\kappa_{\gamma_\varepsilon(t)} - 4\varepsilon\kappa_{\gamma_\varepsilon(t)}\partial_s^3\kappa_{\gamma_\varepsilon(t)} + 4\varepsilon\partial_s\kappa_{\gamma_\varepsilon(t)}\partial_s^2\kappa_{\gamma_\varepsilon(t)}\Big]_0^{\ell(\gamma_\varepsilon(t))} \\
&\quad= \int_0^{\ell(\gamma_\varepsilon(t))} \Big(-2(\partial_s \kappa_{\gamma_\varepsilon(t)})^2 + \kappa^4_{\gamma_\varepsilon(t)}\Big) ds + \varepsilon \int_0^{\ell(\gamma_\varepsilon(t))} \Big(-4(\partial_s^2 \kappa_{\gamma_\varepsilon(t)})^2 - 4\kappa^3_{\gamma_\varepsilon(t)}\partial_s^2\kappa_{\gamma_\varepsilon(t)} - \kappa^6_{\gamma_\varepsilon(t)}\Big) ds \\
&\qquad + \Big[2\kappa_{\gamma_\varepsilon(t)}\partial_s\kappa_{\gamma_\varepsilon(t)} - 4\varepsilon\kappa_{\gamma_\varepsilon(t)}\partial_s^3\kappa_{\gamma_\varepsilon(t)} + 4\varepsilon\partial_s\kappa_{\gamma_\varepsilon(t)}\partial_s^2\kappa_{\gamma_\varepsilon(t)}\Big]_0^{\ell(\gamma_\varepsilon(t))}
\end{aligned} \tag{3.20}$$

where in the last equality, taking also into account the third formula in (1.3), we used that $-3\int_0^{\ell(\gamma_\varepsilon(t))} \kappa^2_{\gamma_\varepsilon(t)}(\partial_s\kappa_{\gamma_\varepsilon(t)})^2 ds = \int_0^{\ell(\gamma_\varepsilon(t))} \kappa^3_{\gamma_\varepsilon(t)}\partial_s^2\kappa_{\gamma_\varepsilon(t)} ds$.
It remains to show that the contribution of the boundary term is zero. From (1.3) we have $\kappa_{\gamma_\varepsilon(t)}(s) = 0, s \in \{0, \ell(\gamma_\varepsilon(t))\}$, thus we only need to show that

$$\big[\partial_s\kappa_{\gamma_\varepsilon(t)}\partial_s^2\kappa_{\gamma_\varepsilon(t)}\big]_0^{\ell(\gamma_\varepsilon(t))} = 0 \qquad \forall t \in (0, +\infty). \tag{3.21}$$

This follows from the fact that, for $s \in \{0, \ell(\gamma_\varepsilon(t))\}$,

$$0 = E_\varepsilon(t, s) = -\kappa_{\gamma_\varepsilon(t)}(s) + \varepsilon\Big(2\partial_s^2\kappa_{\gamma_\varepsilon(t)}(s) + \kappa^3_{\gamma_\varepsilon(t)}(s)\Big) = 2\varepsilon\partial_s^2\kappa_{\gamma_\varepsilon(t)}(s)$$

where we have used (3.3), (3.4), (3.1) and $\kappa_{\gamma_\varepsilon(t)}(s) = 0, s \in \{0, \ell(\gamma_\varepsilon(t))\}$. □

We now recall the following version of the Gagliardo–Nirenberg inequality which follows from [6, Theorem 1] and a scaling argument.

Theorem 3.7 (**Gagliardo-Nirenberg interpolation inequalities**). *Let η be a smooth curve in $\mathbb{R}^2$ with length $L \in (0, +\infty)$ and let u be a smooth function defined on η. Then for every $j \geq 1$, $p \in [2, +\infty]$ and $n \in \{0, \dots, j-1\}$ we have the estimates*

$$||\partial_s^n u||_{L^p} \leq \tilde{C}_{n,j,p}||\partial_s^j u||^\sigma_{L^2}||u||^{1-\sigma}_{L^2} + \frac{B_{n,j,p}}{L^{j\sigma}}||u||_{L^2}$$

where

$$\sigma = \frac{n + 1/2 - 1/p}{j}$$

and the constants $\tilde{C}_{n,j,p}$ and $B_{n,j,p}$ are independent of η. In particular, if $p = +\infty$,

$$||\partial_s^n u||_{L^\infty} \leq \tilde{C}_{n,j}||\partial_s^j u||^\sigma_{L^2}||u||^{1-\sigma}_{L^2} + \frac{B_{n,j}}{L^{j\sigma}}||u||_{L^2} \quad \text{with} \quad \sigma = \frac{n + 1/2}{j}.$$

Remark 3.8. If $n = 0, j = 2, p = 6$ we get $\sigma = 1/6$ and

$$||u||_{L^6} \leq C||\partial_s^2 u||^{\frac{1}{6}}_{L^2}||u||^{\frac{5}{6}}_{L^2} + \frac{C}{L^{\frac{1}{3}}}||u||_{L^2},$$

for some $C > 0$, hence, by means of Young inequality $|xy| \leq \frac{1}{a}|x|^a + \frac{1}{b}|y|^b$, $1/a + 1/b = 1$, choosing $a = b = 2$, $x = \sqrt{2}||\partial_s^2 u||^{1/2}_{L^2}$ and $y = \frac{1}{\sqrt{2}}||u||^{5/2}_{L^2}$, we obtain

$$\int_\gamma u^6 ds \leq \int_\gamma (\partial_s^2 u)^2 ds + C\Big(\int_\gamma u^2 ds\Big)^5 + \frac{C}{L^2}\Big(\int_\gamma u^2 ds\Big)^3. \tag{3.22}$$

If $n = 0, j = 1, p = 4$ we get $\sigma = 1/4$ and

$$||u||_{L^4} \le C||\partial_s u||_{L^2}^{\frac{1}{4}}||u||_{L^2}^{\frac{3}{4}} + \frac{C}{L^{\frac{1}{4}}}||u||_{L^2},$$

hence, reasoning as before,

$$\int_\gamma u^4 ds \le \int_\gamma (\partial_s u)^2 ds + C\Big(\int_\gamma u^2 ds\Big)^3 + \frac{C}{L}\Big(\int_\gamma u^2 ds\Big)^2. \tag{3.23}$$

The next result is proven in [1, Proposition 3.10], and holds also for a curve with boundary. The idea of the proof is to use Lemma 3.6, adding a suitable positive quantity in order to eliminate terms whose sign is not known, and then estimating the resulting expressions using (3.22) and (3.23).

Lemma 3.9. *There exists a positive constant $C > 0$ independent of $\varepsilon \in (0, 1]$ such that the following estimate holds:*

$$\begin{aligned}&\frac{d}{dt}\int_0^{\ell(\gamma_\varepsilon(t))} \kappa^2_{\gamma_\varepsilon(t)}\, ds \\ \le &C\Big(\int_0^{\ell(\gamma_\varepsilon(t))} \kappa^2_{\gamma_\varepsilon(t)} ds\Big)^5 + C\Big(\int_0^{\ell(\gamma_\varepsilon(t))} \kappa^2_{\gamma_\varepsilon(t)} ds\Big)^3 + C\Big(\int_0^{\ell(\gamma_\varepsilon(t))} \kappa^2_{\gamma_\varepsilon(t)} ds\Big)^2.\end{aligned} \tag{3.24}$$

Proof of Proposition 3.5 The statement follows by Lemma 3.9, estimate (3.12) and by the Gronwall's lemma. $\square$

3.3 ε-uniform estimates of $\|\partial_s^j \kappa_{\gamma_\varepsilon}\|_2$

We deal now with the higher derivatives of the curvature, obtaining this crucial result:

Proposition 3.10. *There exists $T_{\max} > 0$ such that, for any $T \in (0, T_{\max})$ and $j \in \mathbb{N}$*

$$\sup_{\varepsilon\in(0,1]} \sup_{t\in[0,T]} \int_0^{\ell(\gamma_\varepsilon(t))} (\partial_s^j \kappa_{\gamma_\varepsilon(t)})^2\, ds < +\infty. \tag{3.25}$$

In order to prove Proposition 3.10 we need some preliminaries.

Definition 3.11. For $l \in \mathbb{N}$, we denote by $\mathfrak{q}^r(\partial_s^l \kappa)$ a polynomial in $\kappa, \dots, \partial_s^l \kappa$ with constant coefficients in $\mathbb{R}$ such that every monomial it contains is of the form

$$\prod_{i=1}^N \partial_s^{j_i}\kappa \qquad \text{with } 0 \le j_i \le l,\ N \ge 1 \text{ and } r = \sum_{i=1}^N (j_i + 1).$$

For the rest of this section all polynomials in the curvature $\kappa_{\gamma_\varepsilon(t)}$ and its derivatives are completely contracted, that is they belong to the family $\mathfrak{q}^r(\partial_s^l \kappa_{\gamma_\varepsilon(t)})$ as defined above.
The following formula is proved in [5, Lemma 2.19]:

Lemma 3.12. *Let $\varepsilon \in (0, 1]$ and $\gamma_\varepsilon : dom(\gamma_\varepsilon) \to \mathbb{R}^2$ be a curve evolving by (3.2). Then, for any $j \in \mathbb{N}$, we have*

$$\begin{aligned}\partial_t\partial_s^j \kappa_{\gamma_\varepsilon(t)} = {}&\partial_s^{j+2}\kappa_{\gamma_\varepsilon(t)} + \mathfrak{q}^{j+3}(\partial_s^j \kappa_{\gamma_\varepsilon(t)}) - 2\varepsilon\partial_s^{j+4}\kappa_{\gamma_\varepsilon(t)} - 5\varepsilon\kappa^2_{\gamma_\varepsilon(t)}\partial_s^{j+2}\kappa_{\gamma_\varepsilon(t)} \\ &+ \varepsilon\mathfrak{q}^{j+5}(\partial_s^{j+1}\kappa_{\gamma_\varepsilon(t)}) + \lambda_\varepsilon\partial_s^{j+1}\kappa_{\gamma_\varepsilon(t)}.\end{aligned} \tag{3.26}$$

From the conditions (see (1.3), (3.3), (3.4))

$$\kappa_{\gamma_\varepsilon(t)} = 0 \qquad \text{at } \{0, \ell(\gamma_\varepsilon(t))\}, \tag{3.27}$$

$$\langle \partial_t \gamma_\varepsilon(t), \nu_{\gamma_\varepsilon(t)} \rangle = E_\varepsilon = 0 \qquad \text{at } \{0, \ell(\gamma_\varepsilon(t))\},$$

and using the expression of E_ε in (3.1) it follows

$$\partial_s^2 \kappa_{\gamma_\varepsilon(t)} = 0 \qquad \text{at } \{0, \ell(\gamma_\varepsilon(t))\}, \quad t \in (0, +\infty). \tag{3.28}$$

Also from the condition $\partial_t \kappa_{\gamma_\varepsilon(t)}(0) = 0$, and from

$$\langle \partial_t \gamma_\varepsilon(t, 0), \tau_{\gamma_\varepsilon(t)}(0) \rangle = \lambda_\varepsilon(t, 0) = 0 \tag{3.29}$$

(see (3.3)), using (3.27), (3.28) and the evolution equation (3.10) it follows

$$\partial_s^4 \kappa_{\gamma_\varepsilon(t)}(0) = 0, \quad t \in (0, +\infty). \tag{3.30}$$

Looking now at $s = \ell(\gamma_\varepsilon(t))$, from the condition

$$0 = \frac{d}{dt} \kappa_{\gamma_\varepsilon(t)}(\ell(\gamma_\varepsilon(t))) = \partial_t \kappa_{\gamma_\varepsilon(t)}(\ell(\gamma_\varepsilon(t))) + \partial_s \kappa_{\gamma_\varepsilon(t)}(\ell(\gamma_\varepsilon(t))) \dot{\ell}(\gamma_\varepsilon(t)),$$

and from the evolution equation (3.10), (3.27), (3.28) and

$$\langle \partial_t \gamma_\varepsilon(t, \ell(\gamma_\varepsilon(t))), \tau_{\gamma_\varepsilon(t)}(\ell(\gamma_\varepsilon(t))) \rangle = \lambda_\varepsilon(t, \ell(\gamma_\varepsilon(t))) = -\dot{\ell}(\gamma_\varepsilon(t)) \tag{3.31}$$

(see (3.5)), we obtain

$$\begin{aligned} 0 =& -2\varepsilon \partial_s^4 \kappa_{\gamma_\varepsilon(t)}(\ell(\gamma_\varepsilon(t))) + \lambda_\varepsilon(t, \ell(\gamma_\varepsilon(t))) \partial_s \kappa_{\gamma_\varepsilon(t)}(\ell(\gamma_\varepsilon(t))) + \partial_s \kappa_{\gamma_\varepsilon(t)}(\ell(\gamma_\varepsilon(t))) \dot{\ell}(\gamma_\varepsilon(t)) \\ =& -2\varepsilon \partial_s^4 \kappa_{\gamma_\varepsilon(t)}(\ell(\gamma_\varepsilon(t))) + \lambda_\varepsilon(t, \ell(\gamma_\varepsilon(t))) \partial_s \kappa_{\gamma_\varepsilon(t)}(\ell(\gamma_\varepsilon(t))) - \lambda_\varepsilon(t, \ell(\gamma_\varepsilon(t))) \partial_s \kappa_{\gamma_\varepsilon(t)}(\ell(\gamma_\varepsilon(t))) \\ =& -2\varepsilon \partial_s^4 \kappa_{\gamma_\varepsilon(t)}(\ell(\gamma_\varepsilon(t))). \end{aligned} \tag{3.32}$$

Combining (3.30) and (3.32) we conclude

$$\partial_s^4 \kappa_{\gamma_\varepsilon(t)} = 0, \qquad \text{at } \{0, \ell(\gamma_\varepsilon(t))\}, \quad t \in (0, +\infty). \tag{3.33}$$

For the following lemma see also [5, Lemma 4.8] and [7, Lemma 2.5].

Lemma 3.13. *For any $t \in (0, +\infty)$*

$$\partial_s^j \kappa_{\gamma_\varepsilon(t)} = 0 \qquad \text{at } \{0, \ell(\gamma_\varepsilon(t))\} \quad \forall j \text{ even}. \tag{3.34}$$

Proof. We write $j = 2n$, and we argue by induction on n. The first step of induction $n = 1$ is given by (3.28). Let $n \in \mathbb{N}$, $n \geq 2$ and suppose that $\partial_s^{2m} \kappa_{\gamma_\varepsilon(t)}(0) = \partial_s^{2m} \kappa_{\gamma_\varepsilon(t)}(\ell(\gamma_\varepsilon(t))) = 0$ holds for any natural number $m \leq n$. We have to show that

$$\partial_s^{2(n+1)} \kappa_{\gamma_\varepsilon(t)} = 0 \qquad \text{at } \{0, \ell(\gamma_\varepsilon(t))\}. \tag{3.35}$$

Using (3.26) with $j = 2(n-1)$, we have, at $s = 0$,

$$\begin{aligned} 0 = \partial_t \partial_s^{2(n-1)} \kappa_{\gamma_\varepsilon(t)} =& \partial_s^{2n} \kappa_{\gamma_\varepsilon(t)} + \mathfrak{q}^{2n+1}(\partial_s^{2n-2} \kappa_{\gamma_\varepsilon(t)}) - 2\varepsilon \partial_s^{2(n+1)} \kappa_{\gamma_\varepsilon(t)} - 5\varepsilon \kappa_{\gamma_\varepsilon(t)}^2 \partial_s^{2n} \kappa_{\gamma_\varepsilon(t)} \\ &+ \varepsilon \mathfrak{q}^{2n+3}(\partial_s^{2n-1} \kappa_{\gamma_\varepsilon(t)}) + \lambda_\varepsilon \partial_s^{2n-1} \kappa_{\gamma_\varepsilon(t)} \\ =& \mathfrak{q}^{2n+1}(\partial_s^{2n-2} \kappa_{\gamma_\varepsilon(t)}) - 2\varepsilon \partial_s^{2(n+1)} \kappa_{\gamma_\varepsilon(t)} + \varepsilon \mathfrak{q}^{2n+3}(\partial_s^{2n-1} \kappa_{\gamma_\varepsilon(t)}), \end{aligned}$$

where we use (3.27) and (3.29), and the induction hypothesis which ensures $\partial_s^{2n} \kappa_{\gamma_\varepsilon(t)}(0) = 0$.

Using again the induction hypothesis and (3.26) with $j = 2(n-1)$, we have, at $s = \ell(\gamma_\varepsilon(t))$,

$$
\begin{aligned}
0 = \frac{d}{dt}\partial_s^{2(n-1)}\kappa_{\gamma_\varepsilon(t)} &= \partial_t\partial_s^{2(n-1)}\kappa_{\gamma_\varepsilon(t)} + \partial_s^{2(n-1)}\kappa_{\gamma_\varepsilon(t)}\dot{\ell}(\gamma_\varepsilon(t)) \\
&= \partial_s^{2n}\kappa_{\gamma_\varepsilon(t)} + \mathfrak{q}^{2n+1}(\partial_s^{2n-2}\kappa_{\gamma_\varepsilon(t)}) - 2\varepsilon\partial_s^{2(n+1)}\kappa_{\gamma_\varepsilon(t)} - 5\varepsilon\kappa^2_{\gamma_\varepsilon(t)}\partial_s^{2n}\kappa_{\gamma_\varepsilon(t)} \\
&\quad + \varepsilon\mathfrak{q}^{2n+3}(\partial_s^{2n-1}\kappa_{\gamma_\varepsilon(t)}) + \lambda_\varepsilon\partial_s^{2n-1}\kappa_{\gamma_\varepsilon(t)} - \lambda_\varepsilon\partial_s^{2n-1}\kappa_{\gamma_\varepsilon(t)} \\
&= \mathfrak{q}^{2n+1}(\partial_s^{2n-2}\kappa_{\gamma_\varepsilon(t)}) - 2\varepsilon\partial_s^{2(n+1)}\kappa_{\gamma_\varepsilon(t)} + \varepsilon\mathfrak{q}^{2n+3}(\partial_s^{2n-1}\kappa_{\gamma_\varepsilon(t)}),
\end{aligned}
$$

where we use also (3.31). Now we observe that

$$
\mathfrak{q}^{2n+1}(\partial_s^{2n-2}\kappa_{\gamma_\varepsilon(t)}) = 0, \quad \mathfrak{q}^{2n+3}(\partial_s^{2n-1}\kappa_{\gamma_\varepsilon(t)}) = 0 \quad \text{at } \{0, \ell(\gamma_\varepsilon(t))\}.
$$

Actually, we shall prove more, namely that each monomial of $\mathfrak{q}^{2n+1}(\partial_s^{2n-2}\kappa_{\gamma_\varepsilon(t)})$ and of $\mathfrak{q}^{2n+3}(\partial_s^{2n-1}\kappa_{\gamma_\varepsilon(t)})$ vanishes at $\{0, \ell(\gamma_\varepsilon(t))\}$. Indeed, recalling Definition 3.11, we have that $\mathfrak{q}^{2n+1}(\partial_s^{2n-2}\kappa_{\gamma_\varepsilon(t)})$ is a polynomial in $\kappa_{\gamma_\varepsilon(t)}, \dots, \partial_s^{2n-2}\kappa_{\gamma_\varepsilon(t)}$ such that every of its monomials is of the form

$$
\prod_{i=1}^{N}\partial_s^{j_i}\kappa_{\gamma_\varepsilon(t)} \qquad \text{with } 0 \le j_i \le 2n-2, \text{ for some } \ N \ge 1
$$

and

$$
2n+1 = \sum_{i=1}^{N}(j_i+1). \tag{3.36}
$$

Now, we observe that it is not possible that all indices j_i, for $i \in \{1, \dots, N\}$ are odd, as a direct consequence of (3.36). Therefore, there exists at least one index $i \in \{1, \dots, N\}$ such that j_i is even. Since $j_i \le 2n-2$, the inductive hypothesis implies that $\partial_s^{j_i}\kappa_{\gamma_\varepsilon(t)} = 0$ at $\{0, \ell(\gamma_\varepsilon(t))\}$. Consequently, the entire monomial vanishes.
Similarly, monomials of $\mathfrak{q}^{2n+3}(\partial_s^{2n-1}\kappa_{\gamma_\varepsilon(t)})$ are of the form

$$
\prod_{i=1}^{N}\partial_s^{j_i}\kappa_{\gamma_\varepsilon(t)} \qquad \text{with } 0 \le j_i \le 2n-1, \text{ for some } N \ge 1
$$

and

$$
2n+3 = \sum_{i=1}^{N}(j_i+1). \tag{3.37}
$$

As in the previous case, it is not possible that all indices j_i, for $i \in \{1, \dots, N\}$ are odd, since this would contradict (3.37). Therefore, there exists at least one index $i \in \{1, \dots, N\}$ such that j_i is even. Since $j_i \le 2n-1$, the inductive hypothesis implies that $\partial_s^{j_i}\kappa_{\gamma_\varepsilon(t)} = 0$ at $\{0, \ell(\gamma_\varepsilon(t))\}$ and the entire monomial vanishes. This completes the proof of (3.35) and (3.34) follows. □

Using Lemma 3.13 we prove that [1, Lemma 3.12] is still valid, since we can show that the boundary terms at any order vanish:

Proposition 3.14. *For any $t \in (0, +\infty)$ and any $j \in \mathbb{N}$ we have*

$$
\begin{aligned}
\frac{d}{dt}\int_{\gamma_\varepsilon(t)}(\partial_s^j\kappa_{\gamma_\varepsilon(t)})^2\,ds = &-2\int_{\gamma_\varepsilon(t)}(\partial_s^{j+1}\kappa_{\gamma_\varepsilon(t)})^2\,ds - 4\varepsilon\int_{\gamma_\varepsilon(t)}(\partial_s^{j+2}\kappa_{\gamma_\varepsilon(t)})^2\,ds \\
&+ \int_{\gamma_\varepsilon(t)}\mathfrak{q}^{2j+4}(\partial_s^j\kappa_{\gamma_\varepsilon(t)})\,ds + \varepsilon\int_{\gamma_\varepsilon(t)}\mathfrak{q}^{2j+6}(\partial_s^{j+1}\kappa_{\gamma_\varepsilon(t)})\,ds\,.
\end{aligned}
$$

Proof. Using (3.7) and (3.26) we deduce, also coupling together the term containing λ_ε in $\partial_t\partial_s^j\kappa_{\gamma_\varepsilon(t)}$ and the term $\partial_s^j\kappa_{\gamma_\varepsilon(t)}\partial_s\lambda_\varepsilon$,

$$\begin{aligned}\frac{d}{dt}\int_0^{\ell(\gamma_\varepsilon(t))}(\partial_s^j\kappa_{\gamma_\varepsilon(t)})^2\,ds &= 2\int_0^{\ell(\gamma_\varepsilon(t))}\partial_s^j\kappa_{\gamma_\varepsilon(t)}\partial_t\partial_s^j\kappa_{\gamma_\varepsilon(t)}\,ds+\int_0^{\ell(\gamma_\varepsilon(t))}(\partial_s^j\kappa_{\gamma_\varepsilon(t)})^2(\kappa_{\gamma_\varepsilon}E_\varepsilon+\partial_s\lambda_\varepsilon)\,ds\\ &\quad+\Big(\partial_s^j\kappa_{\gamma_\varepsilon(t)}(\ell(\gamma_\varepsilon(t)))\Big)^2\dot{\ell}(\gamma_\varepsilon(t))\\ &=2\int_0^{\ell(\gamma_\varepsilon(t))}\partial_s^j\kappa_{\gamma_\varepsilon(t)}(\partial_s^{j+2}\kappa_{\gamma_\varepsilon(t)}+\mathfrak{q}^{j+3}(\partial_s^j\kappa_{\gamma_\varepsilon(t)}))\,ds\\ &\quad+\varepsilon\int_0^{\ell(\gamma_\varepsilon(t))}2\partial_s^j\kappa_{\gamma_\varepsilon(t)}\Big(-2\partial_s^{j+4}\kappa_{\gamma_\varepsilon(t)}-5\kappa_{\gamma_\varepsilon(t)}^2\partial_s^{j+2}\kappa_{\gamma_\varepsilon(t)}+\mathfrak{q}^{j+5}(\partial_s^{j+1}\kappa_{\gamma_\varepsilon(t)})\Big)\,ds\\ &\quad-\int_0^{\ell(\gamma_\varepsilon(t))}(\partial_s^j\kappa_{\gamma_\varepsilon(t)})^2\kappa_{\gamma_\varepsilon(t)}\Big(\kappa_{\gamma_\varepsilon(t)}-2\varepsilon\partial_s^2\kappa_{\gamma_\varepsilon(t)}-\varepsilon\kappa_{\gamma_\varepsilon(t)}^3\Big)\,ds\\ &\quad+\int_0^{\ell(\gamma_\varepsilon(t))}\partial_s(\lambda_\varepsilon(\partial_s^j\kappa_{\gamma_\varepsilon(t)})^2)\,ds+\Big(\partial_s^j\kappa_{\gamma_\varepsilon(t)}(\ell(\gamma_\varepsilon(t)))\Big)^2\dot{\ell}(\gamma_\varepsilon(t)).\end{aligned}$$

Notice that (3.29) and (3.31) imply that the last line of the above expression vanishes, i.e.

$$\lambda_\varepsilon(t,\ell(\gamma_\varepsilon(t)))(\partial_s^j\kappa_{\gamma_\varepsilon(t)}(\ell(\gamma_\varepsilon(t))))^2-\lambda_\varepsilon(t,0)(\partial_s^j\kappa_{\gamma_\varepsilon(t)}(0))^2+\Big(\partial_s^j\kappa_{\gamma_\varepsilon(t)}(\ell(\gamma_\varepsilon(t)))\Big)^2\dot{\ell}(\gamma_\varepsilon(t))=0.$$

Integrating by parts we deduce

$$\begin{aligned}\frac{d}{dt}\int_{\gamma_\varepsilon(t)}(\partial_s^j\kappa_{\gamma_\varepsilon(t)})^2\,ds &= -2\int_{\gamma_\varepsilon(t)}(\partial_s^{j+1}\kappa_{\gamma_\varepsilon(t)})^2\,ds-4\varepsilon\int_{\gamma_\varepsilon(t)}(\partial_s^{j+2}\kappa_{\gamma_\varepsilon(t)})^2\,ds\\ &\quad+\int_{\gamma_\varepsilon(t)}\mathfrak{q}^{2j+4}(\partial_s^j\kappa_{\gamma_\varepsilon(t)})\,ds+\varepsilon\int_{\gamma_\varepsilon(t)}\mathfrak{q}^{2j+6}(\partial_s^{j+1}\kappa_{\gamma_\varepsilon(t)})\,ds\\ &\quad+2\Big[\partial_s^j\kappa_{\gamma_\varepsilon(t)}\partial_s^{j+1}\kappa_{\gamma_\varepsilon(t)}\Big]_0^{\ell(\gamma_\varepsilon(t))}-4\varepsilon\Big[\partial_s^j\kappa_{\gamma_\varepsilon(t)}\partial_s^{j+3}\kappa_{\gamma_\varepsilon(t)}\Big]_0^{\ell(\gamma_\varepsilon(t))}\\ &\quad+4\varepsilon\Big[\partial_s^{j+1}\kappa_{\gamma_\varepsilon(t)}\partial_s^{j+2}\kappa_{\gamma_\varepsilon(t)}\Big]_0^{\ell(\gamma_\varepsilon(t))}.\end{aligned}$$

Now Lemma 3.13 implies

$$\partial_s^j\kappa_{\gamma_\varepsilon(t)}\partial_s^{j+1}\kappa_{\gamma_\varepsilon(t)}=\partial_s^j\kappa_{\gamma_\varepsilon(t)}\partial_s^{j+3}\kappa_{\gamma_\varepsilon(t)}=\partial_s^{j+1}\kappa_{\gamma_\varepsilon(t)}\partial_s^{j+2}\kappa_{\gamma_\varepsilon(t)}=0\qquad\text{at }\{0,\ell(\gamma_\varepsilon(t))\},$$

which ensures that the boundary term vanish. The thesis follows. ☐

The next result is proven in [1, Proposition 3.13], and is also valid in our setting.

Proposition 3.15. *For any $j\in\mathbb{N}$ we have the ε- independent estimate, for $\varepsilon\in(0,1)$,*

$$\frac{d}{dt}\int_0^{\ell(\gamma_\varepsilon(t))}(\partial_s^j\kappa_{\gamma_\varepsilon(t)})^2\,ds\le C\Big(\int_0^{\ell(\gamma_\varepsilon(t))}\kappa_{\gamma_\varepsilon(t)}^2\,ds\Big)^{2j+3}+C\Big(\int_0^{\ell(\gamma_\varepsilon(t))}\kappa_{\gamma_\varepsilon(t)}^2\,ds\Big)^{2j+5}+C\quad(3.38)$$

where the constant C depends only on $1/\ell(\gamma_\varepsilon(t))$.

By means of Propositions 3.5 and 3.15 we have then the following result.

Theorem 3.16. *For any $j\in\mathbb{N}$ there exists a smooth function $Z^j:\mathbb{R}\to(0,+\infty)$ such that*

$$\frac{d}{dt}\int_0^{\ell(\gamma_\varepsilon(t))}(\partial_s^j\kappa_{\gamma_\varepsilon(t)})^2\,ds\le Z^j\Big(\int_0^{\ell(\gamma_\varepsilon(t))}\kappa_{\gamma_\varepsilon(t)}^2\,ds\Big)\tag{3.39}$$

for every $\varepsilon\in(0,1)$.

Proof. The statement follows by Propositions 3.5 and 3.15, since by (3.12) the quantity $1/\ell(\gamma_\varepsilon(t))$ is controlled. $\square$

We are now ready to prove Proposition 3.10.

Proof of Proposition 3.10. From

$$\int_0^{\ell(\gamma_\varepsilon(t))} (\partial_s^j \kappa_{\gamma_\varepsilon(t)})^2 \ ds = \int_0^t \frac{d}{dt} \int_0^{\ell(\gamma_\varepsilon(\tau))} (\partial_s^j \kappa_{\gamma_\varepsilon(\tau)})^2 \ ds \ d\tau + \int_0^{\ell(\overline{\gamma})} (\partial_s^j \kappa_{\overline{\gamma}})^2 \ ds,$$

using (3.39) and the smoothness of $\overline{\gamma}$, we get

$$\int_0^{\ell(\gamma_\varepsilon(t))} (\partial_s^j \kappa_{\gamma_\varepsilon(t)})^2 \ ds \leq \int_0^t Z^j \left(\int_0^{\ell(\gamma_\varepsilon(\tau))} \kappa_{\gamma_\varepsilon(\tau)}^2 \ ds \right) + C,$$

where C is a positive constant independent of ε and depending only on $\overline{\gamma}$. Now we conclude by applying (3.17). $\square$

Remark 3.17. As a direct consequence of Proposition 3.10, we deduce that for every $T \in (0, T_{\max})$ there exists a constant $C > 0$, depending on T and $\overline{\gamma}$, such that

$$\sup_{\varepsilon \in (0,1]} |E_\varepsilon(t,x)| \leq C \quad \forall (t,x) \in [0,T] \times [0,1]. \tag{3.40}$$

Indeed, Proposition 3.10 ensures that all spatial derivatives of the curvature are uniformly bounded in L^2 in $[0,T]$. Therefore, for every $n \in \mathbb{N}$,

$$\|\kappa_{\gamma_\varepsilon}(t,\cdot)\|_{W^{n,2}([0,\ell(\gamma_\varepsilon(t))])} < C(n) \qquad \forall t \in [0,T].$$

Hence, by Sobolev embedding, there exists a constant $\hat{C}(n) > 0$ such that

$$\|\partial_s^n \kappa_{\gamma_\varepsilon}(t,\cdot)\|_{L^\infty([0,\ell(\gamma_\varepsilon(t))])} \leq \hat{C}(n).$$

Finally, recalling the expression of E_ε (see (3.1)), which depends on $\kappa_{\gamma_\varepsilon(t)}$, $\partial_s^2 \kappa_{\gamma_\varepsilon(t)}$, and $\kappa_{\gamma_\varepsilon(t)}^3$, estimate (3.40) directly follows.

Remark 3.18. A further consequence of (3.13) and (3.17) is that there exists a constant $C > 0$ independent of ε such that for any $t \in [0, T_{\max})$ and any $\varepsilon \in (0,1]$,

$$\frac{d}{dt}\ell(\gamma_\varepsilon(t)) \leq C \left(\int_0^{\ell(\gamma_\varepsilon(t))} \kappa_{\gamma_\varepsilon(t)}^2 ds \right)^3 + \frac{C}{\ell(\gamma_\varepsilon(t))} \left(\int_0^{\ell(\gamma_\varepsilon(t))} \kappa_{\gamma_\varepsilon(t)}^2 ds \right)^2 \leq C.$$

Indeed, using equations (3.7), (3.29), (3.31) and integrating by parts we get

$$\begin{aligned}
\frac{d}{dt}\ell(\gamma_\varepsilon(t)) &= \frac{d}{dt} \int_0^{\ell(\gamma_\varepsilon(t))} 1 \ ds = \int_0^{\ell(\gamma_\varepsilon(t))} \Big(E_\varepsilon \kappa_{\gamma_\varepsilon(t)} + \partial_s \lambda_\varepsilon \Big) \, ds \ + \dot{\ell}(\gamma_\varepsilon(t)) \\
&= \int_0^{\ell(\gamma_\varepsilon(t))} \Big(- \kappa_{\gamma_\varepsilon(t)}^2 + 2\varepsilon \kappa_{\gamma_\varepsilon(t)} \partial_s^2 \kappa_{\gamma_\varepsilon(t)} + \varepsilon \kappa_{\gamma_\varepsilon(t)}^4 \Big) \, ds \\
&\quad + \lambda_\varepsilon(t, \ell(\gamma_\varepsilon(t))) - \lambda_\varepsilon(t,0) + \dot{\ell}(\gamma_\varepsilon(t)) \\
&= \int_0^{\ell(\gamma_\varepsilon(t))} \Big(- \kappa_{\gamma_\varepsilon(t)}^2 - 2\varepsilon (\partial_s \kappa_{\gamma_\varepsilon(t)})^2 + \varepsilon \kappa_{\gamma_\varepsilon(t)}^4 \Big) \, ds + 2\varepsilon \Big[\kappa_{\gamma_\varepsilon(t)} \partial_s \kappa_{\gamma_\varepsilon(t)} \Big]_0^{\ell(\gamma_\varepsilon(t))}.
\end{aligned}$$

Using the boundary conditions on $\kappa_{\gamma_\varepsilon(t)}$ in (1.3) and (3.23), we find

$$\begin{aligned}
\frac{d}{dt}\ell(\gamma_\varepsilon(t)) \leq & \int_0^{\ell(\gamma_\varepsilon(t))} \Big(- \kappa_{\gamma_\varepsilon(t)}^2 - 2\varepsilon (\partial_s \kappa_{\gamma_\varepsilon(t)})^2 + \varepsilon (\partial_s \kappa_{\gamma_\varepsilon(t)})^2 \Big) \, ds \\
& + \varepsilon C \left(\int_0^{\ell(\gamma_\varepsilon(t))} \kappa_{\gamma_\varepsilon(t)}^2 ds \right)^3 + \varepsilon \frac{C}{\ell(\gamma_\varepsilon(t))} \left(\int_0^{\ell(\gamma_\varepsilon(t))} \kappa_{\gamma_\varepsilon(t)}^2 ds \right)^2 \\
\leq & \, C \left(\int_0^{\ell(\gamma_\varepsilon(t))} \kappa_{\gamma_\varepsilon(t)}^2 ds \right)^3 + \frac{C}{\ell(\gamma_\varepsilon(t))} \left(\int_0^{\ell(\gamma_\varepsilon(t))} \kappa_{\gamma_\varepsilon(t)}^2 ds \right)^2,
\end{aligned}$$

since $\varepsilon \in (0,1]$. We conclude using (3.12) and (3.17). $\square$

3.4 ε-uniform estimate of the tangential velocity

From the expression (3.11) we obtain

$$\begin{aligned}|\lambda_\varepsilon(t,s)| = \left| -\int_0^s E_\varepsilon \kappa_{\gamma_\varepsilon(t)}\, d\sigma \right| &\le \|E_\varepsilon\|_{L^2([0,\ell(\gamma_\varepsilon(t))])} \|\kappa_{\gamma_\varepsilon(t)}\|_{L^2([0,\ell(\gamma_\varepsilon(t))])} \\ &\le \ell(\gamma_\varepsilon(t)) \|E_\varepsilon\|_{L^\infty([0,\ell(\gamma_\varepsilon(t))])} \|\kappa_{\gamma_\varepsilon(t)}\|_{L^2([0,\ell(\gamma_\varepsilon(t))])}.\end{aligned}$$

Thus, using (3.13), (3.17) and (3.40), we deduce that for every $T \in (0, T_{\max})$ there exists a constant $C > 0$, depending on T and $\overline{\gamma}$, such that

$$\sup_{\varepsilon \in (0,1]} |\lambda_\varepsilon(t,x)| \le C \quad \forall (t,x) \in [0,T] \times [0,1]. \tag{3.41}$$

4 Proof of Theorem 1.1

Let $g_0 > 0$ and $Z : (0,+\infty) \to (0,+\infty)$ be a smooth function, for which we assume

$$\liminf_{x \to \infty} Z(x) > 0,$$

so that the maximal forward solution to the Cauchy problem

$$\begin{cases} g'(t) = Z(g(t)), \\ g(0) = g_0, \end{cases} \tag{4.1}$$

is defined on $[0,a)$, for some $a \in (0,+\infty) \cup \{+\infty\}$. Write $g([0,a)) = [g_0,+\infty)$. Function g is continuous and invertible since $g' > 0$, therefore its inverse $g^{-1} : [g_0,+\infty) \to [0,a)$ is continuous and strictly increasing.

Lemma 4.1 (**Doubling time**). *There exists a continuous function $\Theta : (0,+\infty) \to (0,+\infty)$ such that for every $T \in [0,a)$ it holds $T + \Theta(g(T)) < a$, and for all $t \in [T, T + \Theta(g(T))]$ we have $g(t) \le 2g(T)$.*

Proof. We start to define

$$\Theta(s) := g^{-1}(2s) - g^{-1}(s), \qquad s \in [g_0,+\infty),$$

which a positive continuous function. Fix $T \in [0,a)$ and set $s := g(T)$. By definition of Θ we have $T + \Theta(s) = g^{-1}(2s)$. Hence for any $t \in [T,\, T+\Theta(s)]$ we have $t \in [g^{-1}(s),\, g^{-1}(2s)]$, and by monotonicity of g,

$$g(t) \in [g(g^{-1}(s)),\, g(g^{-1}(2s))] = [s,\, 2s].$$

Thus $g(t) \le 2s = 2g(T)$. This argument shows the existence of a function Θ defined on $[g_0,+\infty)$. Now, for any $n \in \mathbb{N}$, the previous construction with $1/n$ in place of g_0 provides a function Θ_n defined on $[1/n,+\infty)$, and by the uniqueness of the solutions of (4.1) and the explicit expression of g^{-1}, if $m < n$, we have $\Theta_m = \Theta_n$ on $[1/m,+\infty)$. This proves the existence of a function Θ defined on the whole of $(0,+\infty)$ and satisfying the required properties. $\square$

We will now show Theorem 1.1. Recall that $T_{\max}$ is defined in Proposition 3.5 and $T_{\max} = \sup\{T > 0 : \Upsilon_\varepsilon \to \Upsilon \text{ in } C^\infty((0,T] \times [0,1])\}$ and that T_{sing} is the first singularity time for Υ.

Proof of Theorem 1.1. From (3.2), (3.40) and (3.41), we know that for every $T \in (0, T_{\max})$ there exists a constant $C > 0$, independent of ε, such that

$$\left| \frac{\partial \Upsilon_\varepsilon(t,x)}{\partial t} \right| \le |E_\varepsilon(t,x)| + |\lambda_\varepsilon(t,x)| \le C \quad \forall (t,x) \in [0,T] \times [0,1], \quad \forall \varepsilon \in (0,1].$$

Moreover, the constant-speed parametrization $\Upsilon_\varepsilon(t,\cdot)$ over $[0,1]$ of $\gamma_\varepsilon(t,\cdot)$ does not degenerate, due to the lower bound on the length (3.12) (and to the upper bound (3.13)).
Hence, by the Ascoli-Arzelà's theorem in time-space, up to a not relabelled subsequence, the immersions Υ_ε uniformly converge, as $\varepsilon \to 0^+$, to some continuous map $\hat{\Upsilon} : [0,T] \times [0,1] \to \mathbb{R}^2$. From (3.25) we have ε-uniform bounds on $\kappa_{\gamma_\varepsilon(t)}$ and all its derivatives with respect to s, and thus, from equation (1.2), also on all its derivatives with respect to time of any order. Thus the convergence of Υ_ε is in $C^\infty_{\rm loc}((0,T] \times [0,1])$ and passing to the limit in (1.2), (1.3), it follows that $\hat{\Upsilon}$ satisfies (1.6). Since the solution of (1.6) is unique, all the sequence (Υ_ε) converges to $\hat{\Upsilon}$ which hence coincides with Υ. Notice that $\hat{\Upsilon}$ is smooth in $(0,T]$, so that

$$T_{\max} \le T_{\rm sing}.$$

We aim to show $T_{\max} = T_{\rm sing}$, namely that $T_{\max} \ge T_{\rm sing}$. Assume by contradiction that

$$T_{\max} < T_{\rm sing}.$$

Then, by the smoothness of Υ on $(0, T_{\rm sing})$, we have

$$\Upsilon(t,\cdot) \to \Upsilon(T_{\max},\cdot) \text{ in } C^\infty([0,1]) \text{ as } t \to T_{\max}^-. \tag{4.2}$$

Set

$$g_0 := \int_0^{\ell(\overline{\gamma})} \kappa_{\overline{\gamma}}^2 \, ds,$$

and consider the solution g of the Cauchy problem (4.1) with Z given by the right hand side of (3.24), namely

$$Z(p) := Cp^5 + Cp^3 + Cp^2, \qquad p > 0.$$

Let $\Theta : (0,+\infty) \to (0,+\infty)$ be given by Lemma 4.1. By (4.2) and the continuity of Θ we get

$$\lim_{t \to T_{\max}^-} \Theta\left(\int_0^{\ell(\gamma(t))} \kappa_{\gamma(t)}^2 \, ds \right) = \Theta\left(\int_0^{\ell(\gamma(T_{\max}))} \kappa_{\gamma(T_{\max})}^2 \, ds \right) =: \theta > 0.$$

Therefore we can pick

$$t^\star \in [T_{\max} - \theta/4, \, T_{\max})$$

such that

$$\Theta\Big(\int_0^{\ell(\gamma(t^\star))} \kappa_{\gamma(t^\star)}^2 \, ds \Big) \ge \tfrac{2}{3}\theta.$$

Our aim is to show the convergence

$$\Upsilon_\varepsilon(t,\cdot) \to \Upsilon(t,\cdot) \quad \text{in} \quad C^\infty([0,1]) \quad \text{for} \quad t \in [t^\star, t^\star + \theta/2]. \tag{4.3}$$

As $t^\star + \theta/2 \ge T_{\max} - \theta/4 + \theta/2 > T_{\max}$, we will get the contradiction.
By the convergence $\Upsilon_\varepsilon(t^\star,\cdot) \to \Upsilon(t^\star,\cdot)$ in $C^\infty([0,1])$ as $\varepsilon \to 0^+$ we have

$$\lim_{\varepsilon \to 0^+} \int_0^{\ell(\gamma_\varepsilon(t^\star))} \kappa_{\gamma_\varepsilon(t^\star)}^2 \, ds = \int_0^{\ell(\gamma(t^\star))} \kappa_{\gamma(t^\star)}^2 \, ds.$$

Hence there exists $\overline{\varepsilon} > 0$ such that

$$\Theta\Big(\int_0^{\ell(\gamma_\varepsilon(t^\star))} \kappa_{\gamma_\varepsilon(t^\star)}^2 \, ds \Big) \ge \tfrac{\theta}{2} \qquad \forall \varepsilon \in (0, \overline{\varepsilon}]. \tag{4.4}$$

Let us now define

$$h_\infty := \sup_{\varepsilon \in (0,\overline{\varepsilon}]} \int_0^{\ell(\gamma_\varepsilon(t^\star))} \kappa_{\gamma_\varepsilon(t^\star)}^2 \, ds, \tag{4.5}$$

which is finite. Continuity of Θ and (4.4) imply

$$\Theta(h_\infty) \geq \tfrac{\theta}{2}. \tag{4.6}$$

Set

$$T_\infty := g^{-1}(h_\infty) > 0.$$

By (4.6) we know $\Theta(g(T_\infty)) = \Theta(h_\infty) \geq \theta/2$, so that

$$[t^\star,\ t^\star + \Theta(g(T_\infty))) \supseteq [t^\star,\ t^\star + \theta/2]. \tag{4.7}$$

Recall (see (3.39) with $j = 0$ and $Z = Z^0$) that

$$\partial_t\Big(\int_0^{\ell(\gamma_\varepsilon(t))} \kappa^2_{\gamma_\varepsilon(t)}\, ds\Big) \leq Z\Big(\int_0^{\ell(\gamma_\varepsilon(t))} \kappa^2_{\gamma_\varepsilon(t)}\, ds\Big) \qquad \forall \varepsilon \in (0,1], \quad \forall t > 0. \tag{4.8}$$

By the definition of h_∞ in (4.5) we have

$$\int_0^{\ell(\gamma_\varepsilon(t^\star))} \kappa^2_{\gamma_\varepsilon(t^\star)}\, ds \leq h_\infty \qquad \forall \varepsilon \in (0, \overline{\varepsilon}]. \tag{4.9}$$

If $\tau \in [t^\star,\ t^\star + \Theta(h_\infty))$ then

$$\tau - (t^\star - T_\infty) \in [T_\infty,\ T_\infty + \Theta(h_\infty)) \cap [0, a).$$

Define

$$\sigma(\tau) := g\big(\tau - (t^\star - T_\infty)\big) \qquad \forall \tau \in [t^\star,\ t^\star + \Theta(h_\infty)).$$

Then

$$\begin{cases} \sigma'(\tau) = g'(\tau - (t^\star - T_\infty)) = Z\big(g(\tau - (t^\star - T_\infty))\big) = Z(\sigma(\tau)), \\ \sigma(t^\star) = g(T_\infty) = h_\infty. \end{cases} \tag{4.10}$$

By (4.9) we have

$$\int_0^{\ell(\gamma_\varepsilon(t^\star))} \kappa^2_{\gamma_\varepsilon(t^\star)}\, ds \leq \sigma(t^\star) \qquad \forall \varepsilon \in (0, \overline{\varepsilon}]. \tag{4.11}$$

Therefore, applying the comparison principle for ODEs in $[t^\star, t^\star + \Theta(h_\infty))$ between:

- the function $t \in (0, +\infty) \mapsto \int_0^{\ell(\gamma_\varepsilon(t))} \kappa^2_{\gamma_\varepsilon(t)}\, ds$, which satisfies (4.8) (hence is a subsolution of (4.10)), and
- σ, solution of (4.10) with the same initial condition at $\tau = t^\star$,

we obtain: for every $\varepsilon \in (0, \overline{\varepsilon}]$ and every $\tau \in [t^\star,\ t^\star + \Theta(h_\infty))$,

$$\int_0^{\ell(\gamma_\varepsilon(\tau))} \kappa^2_{\gamma_\varepsilon(\tau)}\, ds \leq \sigma(\tau) = g\big(\tau - (t^\star - T_\infty)\big). \tag{4.12}$$

Combining (4.12) with the bound

$$g(t) \leq 2g(T_\infty) \qquad \text{for every } t \in [T_\infty,\ T_\infty + \Theta(h_\infty)) \cap [0, a)$$

given by Lemma 4.1 (which holds for the argument $\tau - (t^\star - T_\infty) \in [T_\infty, T_\infty + \Theta(h_\infty)))$ yields

$$\int_0^{\ell(\gamma_\varepsilon(\tau))} \kappa^2_{\gamma_\varepsilon(\tau)}\, ds \leq g\big(\tau - (t^\star - T_\infty)\big) \leq 2g(T_\infty) = 2h_\infty \qquad \forall \varepsilon \in (0, \overline{\varepsilon}],\ \ \forall \tau \in [t^\star,\ t^\star + \Theta(h_\infty)). \tag{4.13}$$

Hence, according to (4.7), inequality (4.13) holds also on the time interval $[t^\star,\ t^\star + \theta/2]$. Arguing as in the proof of Proposition 3.10, we deduce uniform bounds for all $\|\partial_s^j \kappa_{\gamma_\varepsilon(\tau)}\|_{L^2}$, for every $j \in \mathbb{N}$, in the same time interval. This yields (4.3), and this gives the contradiction. When $\overline{\gamma} \in C^\infty([0, \ell(\overline{\gamma})])$ and all derivatives of $\overline{\gamma}$ of even order at 0 and $\ell(\overline{\gamma})$ vanish, all previous estimates are valid including the initial time $t = 0$, and the convergence takes place in $C^\infty_{\mathrm{loc}}([0, T_{\mathrm{sing}}) \times [0, 1])$. □

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